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Long-Time Integration of Nonlinear Wave Equations with Neural Operators

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Abstract. Neural operators have shown promise in solving many types of Partial Differential Equations (PDEs). They are significantly faster compared to traditional numerical solvers once they have been trained with a certain amount of observed data. However, their numerical performance in solving time-dependent PDEs, particularly in long-time prediction of dynamic systems, still needs improvement. In this paper, we focus on solving the long-time integration of nonlinear wave equations via neural operators by replacing the initial condition with the prediction in a recurrent manner. Given limited observed temporal trajectory data, we utilize some intrinsic features of these nonlinear wave equations, such as conservation laws and well-posedness, to improve the algorithm design and reduce accumulated error. Our numerical experiments examine these improvements in the Korteweg-de Vries (KdV) equation, the sine-Gordon equation, and the Klein-Gordon wave equation on the irregular domain.

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Key words: Neural operator, nonlinear wave equations, long-time integration.

1 Introduction

Solving Partial Differential Equations (PDEs) is central in many physical and engineering fields. Deep learning methodologies and data-driven architectures have become potential alternatives to numerical solvers, considering their low computational costs compared with traditional numerical methods. Among them, neural operators are one

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class of cutting-edge algorithms. Neural operators learn mappings between infinite-dimensional function spaces, which can be exploited to model the solution operators mapping the parameter functions to solutions, the boundary conditions to solutions, and the initial states to solution trajectories in time-dependent problems. The architectures of neural operators include, but are not limited to, Deep Operator Net (DeepONet) [25], Fourier Neural Operator (FNO) [15,22], PCA-Net [3], Graph Neural Operator (GNO) [20], multiwavelet-based model (MWT) [11], Wavelet Neural Operator (WNO) [38], Operator Transformer (OFormer) [18], and General Neural Operator Transformer (GNOT) [12].

Neural operators have shown advantages compared with conventional methods. Once trained, neural operators can predict the solution given any input function through a network forward calculation process, which is usually a composition of matrix multiplication operations and thus saves computation time compared with traditional numerical methods. Neural operator models are usually meshless methods and resolution-invariant because they can make zero-shot super-resolution predictions: trained on a low-resolution dataset, and give predictions on a high-resolution grid or point cloud. Hence, these models indeed output infinite high-resolution predictions (functions) and learn the operator structure between infinite-dimensional function spaces, which distinguishes them from many traditional methods limited to a mesh structure and constant resolution.

While previous works have applied neural operators to solving various PDE problems, few have considered long-time dynamic system modeling via neural operator methods, especially predicting the long-time integration of nonlinear wave equations. To predict the solution u over a temporal interval $[0,T]$ given initial condition u_0, an evident and effective method is training a one-step operator mapping u_0 to the complete solution trajectory. However, this method often comes at the price of acquiring long-time observed data. When the length of the data trajectory is significantly insufficient compared to the required prediction time T, a natural remedy is to train a short-term predictor and recurrently apply itself to its prediction, which is seen as the next-step initial condition. Although theoretically feasible, many fundamental difficulties will be encountered in numerical implementation. For one-step prediction, the neural network method sacrifices a small amount of accuracy in exchange for a multiple reduction in computation time. This trade-off is attractive and fairly acceptable. However, the error may rise sharply when applying the neural operator to an iterative prediction process. The predictions usually become completely inaccurate, unstable, and blow up in several steps or even the second step. The nonlinearity of the wave equation also introduces nonlinear phenomena, including solitons, shocks, rough waves, peakons, and cnoidal waves. These nonlinear waves usually occur in a long-time evolution, making it difficult to capture by short-term predictors and making this problem more challenging.

However, considering some regularity properties, such as the well-posedness and conservation laws of the nonlinear wave equations, we may expect a numerical stable model that approximately adheres to these properties with linearly growing accumulated error. Regardless of some existing long-time integration models combining neural oper-