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Maximum-Principle-Preserving and Positivity-Preserving Central WENO Schemes on Overlapping Meshes

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Abstract. In this paper, we propose a class of maximum-principle-preserving central WENO schemes for scalar conservation laws, and positivity-preserving central WENO schemes for compressible Euler equations. Formulated in a finite volume framework on overlapping meshes, the central schemes require neither flux splitting nor numerical fluxes that are often exact or approximate Riemann solvers. A new fifth-order WENO reconstruction is applied for the spatial discretization, and the linear weights of such reconstruction can be any positive number as long as their sum equals one, which leads to much simpler implementation. The sufficient conditions are provided for the cell average values to preserve maximum principle or positivity property with Euler forward time discretization. The method can be generalized to high-order strong stability preserving Runge-Kutta method without technical difficulties. Extensive numerical examples are presented to illustrate the accuracy and performance of the proposed methods.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Maximum-principle-preserving, positivity-preserving, central scheme, WENO reconstruction, overlapping mesh.

1 Introduction

In this paper, we are interested in one- and two-dimensional hyperbolic conservation laws

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$$\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \end{cases} \tag{1.1}$$

with suitable initial and boundary conditions. Here Eq. (1.1) can be scalar or a system, and it is often nonlinear. The solution to the scalar conservation law has a maximum-principle-preserving (MPP) property such that, if the initial value is bounded $m \leq u(\mathbf{x},0) \leq M$, then the solution is also bounded $m \leq u(\mathbf{x},t) \leq M$ for $t > 0$. Similarly, the solutions to the compressible Euler system have a positivity-preserving (PP) property such that both density and pressure should keep positive in every situation.

Solutions outside of $[m,M]$ might be meaningless, such as probability distribution larger than one or negative percentage. Furthermore, negative density or pressure in gas dynamics equations may lead to instability of system, and this explains that the failure of preserving the positivity of density or pressure may cause blow-ups in the numerical simulations. The E-schemes (which are entropy stable with respect to all convex entropies) such as Godunov, Lax-Friedrichs as well as Engquist-Osher methods are total-variation-diminishing (TVD) and thus MPP. However, any TVD scheme is at most first-order around smooth extrema. The first- and second-order positivity-preserving schemes were designed in [7,15]. For the successful high-order methods, such as Runge-Kutta discontinuous Galerkin (RKDG) and weighted essentially non-oscillatory (WENO) schemes, many efforts have been made to satisfy MPP or PP properties for the robustness and the physical relevance of the numerical solutions. In [28], Zhang and Shu proposed MPP finite volume WENO and DG schemes for scalar conservation laws. A scaling limiter was constructed and applied to the reconstructed polynomials without destroying the local conservation and accuracy. This technique was further extended to preserve the positivity of density and pressure for compressible Euler equations [29–31]. In [23], Wang et al. proposed a simple and robust strategy for the PP DG schemes, and this strategy is also adopted in our paper. To improve the compactness of the stencil of spatial reconstruction, the PP Hermite WENO (HWENO) schemes were designed in [2,8]. Different from the above procedure, Xu developed a parametrized MPP technique by limiting the high-order numerical fluxes toward first-order monotone fluxes in a conservative scheme in the framework of Flux Corrected Transport method [25]. Later, the flux limiter was further generalized to PP finite difference and finite volume WENO schemes for compressible Euler equations [5,24]. More related work can be referred to [9,27].

Compared to upwind type schemes, central schemes are a family of efficient methods for hyperbolic conservation laws. Central methods require neither flux splitting nor numerical fluxes that are often exact or approximate Riemann solvers. In 1990, Nessyahu and Tadmor first proposed a second-order central scheme [18]. With the success of [18], various high-order versions of central schemes were explored, such as central WENO or HWENO methods [10,12,19,21,22] and central DG methods [17]. In addition, MPP or PP central DG methods were developed to solve hyperbolic conservation laws [14], MHD equations [4,6] and shallow water equations [13].