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A Locally Mass-Conservative Enriched Petrov-Galerkin Method Without Penalty for the Darcy Flow in Porous Media

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Abstract. In this work we present an enriched Petrov-Galerkin (EPG) method for the simulation of the Darcy flow in porous media. The new method enriches the approximation trial space of the conforming continuous Galerkin (CG) method with bubble functions and enriches the approximation test space of the CG method with piecewise constant functions, and it does not require any penalty term in the weak formulation. Moreover, we propose a framework for constructing the bubble functions and consider a decoupled algorithm for the EPG method based on this framework, which enables the process of solving pressure to be decoupled into two steps. The first step is to solve the pressure by the standard CG method, and the second step is a post-processing correction of the first step. Compared with the CG method, the proposed EPG method is locally mass-conservative, while keeping fewer degrees of freedom than the discontinuous Galerkin (DG) method. In addition, this method is more concise in the error analysis than the enriched Galerkin (EG) method. The coupled flow and transport in porous media is considered to illustrate the advantages of locally mass-conservative properties of the EPG method. We establish the optimal convergence of numerical solutions and present several numerical examples to illustrate the performance of the proposed method.

AMS subject classifications: 65M60, 65N30, 76S05

Key words: Enriched Petrov-Galerkin method, local mass conservation, post-processing, error analysis.

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1 Introduction

The standard continuous Galerkin (CG) finite element method has been widely applied for the solution of second-order elliptic problems for its simplicity in implementation. However, the CG method possesses limitations in local mass conservation, which is significant in numerous applications, particularly when the coupling of flow and transport is considered [21,48]. Indeed, violation of local mass conservation in velocity fields could cause spurious sources and sinks to the transport simulations, which might lead to numerical inaccuracy for the solution of transport equations.

Lots of numerical methods preserving local mass conservation have been proposed in the literature, which include the mixed finite element method [1–4,46], the finite volume method [12,14,23,26,34], the discontinuous Galerkin (DG) [5,8,15,20,25,28,36,42] method, the recent weak Galerkin method [32], the mixed virtual element method [22] and the reduced basis method [10]. Besides, many other different finite element schemes based on the DG method have been derived, such as the local discontinuous Galerkin method [13, 18], the symmetric interior penalty Galerkin method [41,45], the nonsymmetric interior penalty Galerkin method [37,39], the incomplete interior penalty Galerkin method [21,33, 41], the Oden-Babuška-Baumann-DG method [35,44] and the hybridizable discontinuous Galerkin method [20]. They can tackle rough coefficient cases and capture the nonsmooth features of the solution very well since the function space itself is naturally discontinuous. However, the main disadvantage of the primal DG method is that it has a very large number of degrees of freedom, which can result in expensive computational cost.

In addition, quite a few works have also constructed alternative processing strategies to preserve the local mass conservation [16,19,24,27,43]. Among these methods, the enriched Galerkin (EG) method [29–31,40] precisely enriches the CG finite element space with piecewise constant functions, which has fewer degrees of freedom than the DG method while maintaining local mass conservation. Due to the fact that the EG method is essentially a special DG method, its theoretical analysis is relatively similar to the DG method.

In this paper, our objective is to propose a new enriched Petrov-Galerkin (EPG) method for the simulation of the Darcy flow in porous media. The novelties of this work are as follows:

(I) We design a finite element method which enriches the approximation trial space for the CG method with bubble functions and enriches the approximation test space for the CG method with piecewise constant functions respectively. The main difference between the proposed enrich Petrov-Galerkin (EPG) method and the EG method is the selection of the trial functions. Since the bubble function is zero on the boundary of each element, the trial function is still continuous. Unlike the DG method or the EG method, we can still make use of the analysis framework for the conforming finite element method, which greatly simplifies the analysis compared with the DG and EG methods. Meanwhile, the EPG method and the EG method have the same degree of freedom because the degree of