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Local Discontinuous Galerkin Method for Nonlinear BSPDEs of Neumann Boundary Conditions with Deep Backward Dynamic Programming Time-Marching

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Abstract. This paper aims to present a local discontinuous Galerkin (LDG) method for solving nonlinear backward stochastic partial differential equations (BSPDEs) with Neumann boundary conditions. We establish the L^2-stability and optimal error estimates of the proposed numerical scheme. Two numerical examples are provided to demonstrate the performance of the LDG method, where we incorporate a deep learning algorithm to address the challenge of the curse of dimensionality in backward stochastic differential equations (BSDEs). The results show the effectiveness and accuracy of the LDG method in tackling BSPDEs with Neumann boundary conditions.

AMS subject classifications: 65M60, 65M15, 65Z05

Key words: Local discontinuous Galerkin method, backward stochastic partial differential equations, Neumann boundary problems, stability analysis, error estimates, deep learning algorithm.

1 Introduction

Let $(\Omega,\bar{\mathscr{F}},\{\bar{\mathscr{F}}_t\}_{t\geq 0},\mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. The filtration $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ is generated by two independent d-dimensional Wiener processes W and B. We denote by $\{\mathscr{F}_t\}_{t\geq 0}$ the natural filtration generated by W, together with all $\mathbb{P}$-null sets. The terminal time T is a fixed positive number. The predictable σ-algebra on $\Omega\times[0,T]$ associated with $\{\mathscr{F}_t\}_{t\geq 0}$ and $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ is denoted by $\mathscr{P}$ and $\bar{\mathscr{P}}$, respectively.

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In this paper we present a local discontinuous Galerkin (LDG) method for the following backward stochastic partial differential equations (BSPDEs) with Neumann boundary conditions:

$$\begin{cases} -du(x,t)=\left[\frac{1}{2}\left(|\sigma|^2+|\bar{\sigma}|^2\right)u_{xx}+\sigma\psi_x+\Gamma(\cdot,u,u_x,\psi)\right](x,t)dt-\psi(x,t)dW_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in[0,b]\times[0,T), \\ u_x(0,t)=g(0,t), \quad u_x(b,t)=g(b,t), \qquad t\in[0,T], \\ u(x,T)=G(x), \qquad x\in[0,b], \end{cases} \tag{1.1}$$

which is associated to the following stochastic control problem:

$$\min_{\theta}\mathbb{E}\left[\int_0^T f(t,X_t,\theta_t)dt+\int_0^T g(t,X_t)dL_t+\int_0^T g(t,X_t)dU_t+G(X_T)\right], \tag{1.2}$$

subject to

$$\begin{cases} dX_t=\beta(t,X_t,\theta_t)dt+\sigma(t,X_t)dW_t+\bar{\sigma}(t,X_t)dB_t+dL_t-dU_t, \quad t\in(0,T], \\ X_0=x, \quad L_0=U_0=0, \\ 0\leq X_t\leq b, \quad \text{a.s.}, \\ \int_0^T X_t dL_t=\int_0^T (b-X_t)dU_t=0, \quad \text{a.s.}, \end{cases} \tag{1.3}$$

where L,U are two increasing stochastic processes and $\theta:\Omega\times[0,T]\to\mathcal{U}$ is an admissible control with $\mathcal{U}$ being the control domain, which is a stochastic process. Extensive research efforts have been devoted to Cauchy problems and Dirichlet boundary value problems for BSPDEs. However, the study of Neumann boundary problems for BSPDEs (1.1) remains comparatively underdeveloped, especially in the numerical methods. The only existing work is [2] in which the existence and uniqueness of strong solution for BSPDE with Neumann boundary conditions has been established, which is motivated by the above optimal control problem (1.2)-(1.3) of one-dimensional reflected stochastic differential equations (SDEs) with path-dependent coefficients. With the help of the sufficiently regular solution to (1.1), the optimal feedback control strategies for the associated stochastic control system can be formulated.

Backward stochastic partial differential equations (BSPDEs), as the infinite-dimensional version of backward stochastic differential equations (BSDEs), have been extensively used in problems related to probability theory and stochastic processes, for instance, in the optimal control problems of partial information SDEs or of parabolic stochastic partial differential equations (SPDEs), they appear as dual equations used to construct the stochastic maximum principle (see for example [4, 27, 30]). The stochastic Hamilton-Jacobi-Bellman equations are a special type of nonlinear BSPDEs, first proposed by Peng in [21] in order to study the non-Markovian stochastic control problems. The relation between the forward backward stochastic differential equations (FBSDEs) with random coefficients and BSPDEs which can be viewed as the stochastic Feynman–Kac formula has been established, see for example [14]. The linear, semilinear even