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The First- and Second-Order Energy Stable, Mass Conservative and Bounds-Preserving Schemes for Two-Phase Incompressible Flow with Rock Compressibility

Yujing Yan[1], Huangxin Chen[2] and Xiaoli Li[3,*]

[1] *School of Mathematics, Shandong University, Jinan, Shandong, 250100, P.R. China.*
[2] *School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Xiamen, Fujian, 361005, P.R. China.*
[3] *School of Mathematics and State Key Laboratory of Cryptography and Digital Economy Security, Shandong University, Jinan, Shandong, 250100, P.R. China.*

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Abstract. In this paper, we construct a unified framework of first- and second-order schemes in time for thermodynamically consistent modeling of two-phase incompressible and immiscible flow in porous media with rock compressibility. We rigorously prove that the proposed schemes can preserve the energy dissipation law, conserve the mass of each phase as well as pore volumes and are bounds-preserving for both phases without any restrictions on the time step size. Moreover, the proposed schemes can achieve local mass conservation for both phases and preserve the pore volumes when the saturation of the whole region is within the bounds, and we only need to solve one linear system and several linear algebraic equations, whereas for points where saturation is outside the bounds, we just need to solve an additional nonlinear algebraic equation and hence the proposed schemes can achieve global mass conservation. The developed schemes exhibit high efficiency owing to a substantial reduction in the scale of nonlinear computations and their straightforward implementation. The key point is that we propose a new Lagrange multiplier method that is based on the judgment of saturation bound. This addresses the limitation of the classical Lagrange multiplier method, which is only capable of ensuring global mass conservation. Finally, a variety of illustrative numerical examples including several benchmark problems are provided to validate the accuracy and efficiency of the proposed schemes.

AMS subject classifications: 65M06, 76S05

Key words: Two-phase flow, energy evolution equation, Lagrange multiplier, rock compressibility.

*Corresponding author. *Email addresses:* `yyjsust@163.com` (Y. Yan), `chx@xmu.edu.cn` (H. Chen), `xiaolimath@sdu.edu.cn` (X. Li)

1 Introduction

The incompressible and immiscible two-phase flow in porous media plays a particularly important role in the fields of geology, geotechnical engineering and petroleum reservoir engineering [5, 8, 15], where the mathematical models basically consist of the mass conservation equation and Darcy's law, saturation conditions for the phases and the capillary pressure equation [1,4,24]. The fact that the rock is not compressible has been taken into account in most of the work. However, in unconventional oil and gas production, changes in porosity are important and must be carefully considered. For instance, in certain oil reservoirs, the rock particles have relatively loose structures that lead to large changes in pore volume when the pore fluid pressure fluctuates, thereby leading to the non-negligible changes in rock properties including porosity and permeability [3, 11]. It is crucial to acknowledge that an increase in subsurface material content is typically associated with substantial alterations within the pore spaces and fluid conduits. Thus the rock compressibility should be considered [22, 32, 38].

It is widely acknowledged that the second law of thermodynamics is a crucial factor in the development of effective models. As a comprehensive and fundamental principle, it provides insight into the behaviour of dynamic processes and is an invaluable tool for the development of models in accordance with thermodynamic principles, addressing a diverse range of scientific and engineering challenges [21, 22]. A significant attribute of the model derived from the second law of thermodynamics is that it satisfies the law of energy dissipation. Hence it is important to design the numerical schemes satisfying the discrete energy dissipation law. In general, the fully explicit time discrete schemes preserve energy stability only under severe time step restrictions [9, 14]. The convex-splitting approach [12, 13] usually leads to nonlinear schemes and the linear stabilization approach [31, 37] highly depends on the appropriate choices of stabilization parameters. In recent years, several auxiliary variable approaches, such as the invariant energy quadratization (IEQ) approach and the scalar auxiliary variable (SAV) approach, have been proposed and successfully applied to develop efficient energy stable numerical schemes for the diffuse-interface models [30, 35]. Recently, Wang and He [34] proposed three linearly decoupled and energy stable numerical schemes for compressible multi-component two-phase flows with Peng-Robinson equation of state, incorporated with the multi-component Navier boundary condition. Kou et al. [24] proposed the invariant energy quadratization approach to construct the linear and energy stable numerical method for two-phase incompressible flow with strict time step restrictions. They [18, 21–23] also have constructed several numerical schemes for incompressible, immiscible two-phase flow, single-phase gas flow and multicomponent gas flow in porous media with rock compressibility by using the energy factorization approach. Li et al. [29] proposed high-order and energy stable numerical schemes based on the modified generalized scalar auxiliary variable (mGSAV) approach with new relaxation. In addition to energy stability, the mass conservation law and the bound of variables are crucial for ensuring physically reasonable solutions. The popular methods for constructing bound-preserving schemes