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Exact Boundary Conditions for Periodic Waveguides Containing a Local Perturbation

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Abstract. We consider the solution of the Helmholtz equation $-\Delta u(\mathbf{x}) - n(\mathbf{x})^2\omega^2 u(\mathbf{x}) = f(\mathbf{x})$, $\mathbf{x} = (x,y)$, in a domain Ω which is infinite in x and bounded in y. We assume that $f(\mathbf{x})$ is supported in $\Omega^0 := \{\mathbf{x} \in \Omega \,|a^- < x < a^+\}$ and that $n(\mathbf{x})$ is x-periodic in $\Omega\backslash\Omega^0$. We show how to obtain exact boundary conditions on the vertical segments, $\Gamma^- := \{\mathbf{x} \in \Omega \,|x = a^-\}$ and $\Gamma^+ := \{\mathbf{x} \in \Omega \,|x = a^+\}$, that will enable us to find the solution on $\Omega^0 \cup \Gamma^+ \cup \Gamma^-$. Then the solution can be extended in Ω in a straightforward manner from the values on Γ^- and Γ^+. The exact boundary conditions as well as the extension operators are computed by solving local problems on a single periodicity cell.

Key words: Exact boundary conditions; periodic media; Dirichlet to Neumann maps.

1 Introduction

Periodic media play a major role in applications, in particular in optics for micro and nano-technology. From the point of view of applications, one of the main interesting features is the possibility offered by such media of selecting ranges of frequencies for which waves can or cannot propagate. Mathematically, this property is linked to the gap structure of the spectrum of the underlying differential operator appearing in the model. For a complete, mathematically oriented presentation, we refer the reader to [14, 15].

There is a need for efficient numerical methods for computing the propagation of waves inside such structures. In real applications, the media are not perfectly periodic but differ from periodic media only in bounded regions (which are small with respect to the total size of the propagation domain). In this case, a natural idea is to reduce the pure numerical computations to these regions and to try to take advantage of the periodic structure of

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the problem outside: this is particularly of interest when the periodic regions contain a large number of periodicity cells.

This paper is a contribution to the construction of such methods in a particular situation. We are interested in propagation media which are a local perturbation of an infinite (or very large) periodic waveguide, namely an infinite structure which is periodic in one privileged direction (the propagation direction) and bounded in the other transverse variables (one says that one has a closed waveguide, as opposed to open waveguides as considered in [23], for instance). Physically the perturbation may be a defect or simply a junction. We investigate the question of finding artificial (but exact) boundary conditions to reduce the numerical computation to a neighborhood of this perturbation.

In the case of "classical" waveguides, which are invariant in the propagation direction, (in some sense, periodic with any period), the usual approach consists in applying Dirichlet to Neumann conditions [10, 16] : using the separation of variables, the solution in the semi-infinite waveguide can be written as the superposition of guided modes, that are exponentially varying along the waveguide direction. As a consequence, one can write explicitly a diagonal form of the DtN map in an appropriate (orthonormal) basis. An alternative approach has been proposed recently which uses the method of Perfectly Match Layer (see [2] for the application to waveguides), which does not easily extend to periodic waveguides.

We investigate in this paper the generalization of the DtN approach to periodic waveguides, which is complicated by the fact that separation of variables can no longer be used. However, the notion of guided modes has a natural extension: the notion of Floquet modes. By revisiting the Floquet-Bloch theory [13], we propose a method for constructing DtN operators by solving local problems on a single periodicity cell. This is closely connected to operator-valued Riccati equations (here, of stationary nature), a topic which is already present in many problems concerning artificial boundary conditions (see, for instance, [5, 11]). It appears also that our method is similar to the matrix transfer approach developed for ordinary equations with periodic coefficients [17]. However, except in the 1D case ([8, 19]), this theory cannot be applied directly to our problem due to the fact that the Cauchy problem for the Helmholtz equation is ill-posed in higher dimensions.

In this first paper, our goal is to present the essential ideas and the main theoretical foundations of our method, explain how we can implement it numerically, and present the first numerical results. We expect to give a more complete theory in a subsequent article.

It seems that there are very few works in this direction in the mathematical literature. Most of the existing work is devoted to the determination of the band gap structure of periodic media. The notion of DtN maps did appear, for instance, in the works by T. Abboud [1] for diffraction problems by periodic gratings and by J. Tausch [23] for periodic open waveguides, but it was used to deal with the unboundedness of the propagation medium in the direction(s) transverse to the periodicity direction(s), a much more standard situation than the one we consider in this paper.