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Interaction of Solitary Waves with a Phase Shift in a Nonlinear Dirac Model

Sihong Shao and Huazhong Tang*

LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China.

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Abstract. This paper presents a further numerical study of the interaction dynamics for solitary waves in a nonlinear Dirac model with scalar self-interaction, the Soler model, by using a fourth order accurate Runge-Kutta discontinuous Galerkin method. The phase plane method is employed for the first time to analyze the interaction of Dirac solitary waves and reveals that the relative phase of those waves may vary with the interaction. In general, the interaction of Dirac solitary waves depends on the initial phase shift. If two equal solitary waves are in-phase or out-of-phase initially, so are they during the interaction; if the initial phase shift is far away from 0 and π, the relative phase begins to periodically evolve after a finite time. In the interaction of out-of-phase Dirac solitary waves, we can observe: (a) full repulsion in binary and ternary collisions, depending on the distance between initial waves; (b) repulsing first, attracting afterwards, and then collapse in binary and ternary collisions of initially resting two-humped waves; (c) one-overlap interaction and two-overlap interaction in ternary collisions of initially resting waves.

AMS subject classifications: 65M60, 35L60, 81Q05

Key words: Discontinuous Galerkin method, phase plane method, Dirac field, Soler model, solitary waves, phase shift.

1 Introduction

Ever since its invention in 1929 the Dirac equation has played a fundamental role in various areas of modern physics and mathematics, and is important for the description of interacting particles and fields. Soler [20] first proposed a classical spinorial model with scalar self-interaction for extended particles by means of nonlinear Dirac (NLD)

*Corresponding author. *Email addresses:* `shaosihong@gmail.com` (S. H. Shao), `hztang@math.pku.edu.cn` (H. Z. Tang)

fields. His model is described by the Lagrangian density $L = \mathrm{i}\overline{\boldsymbol{\psi}}\gamma^\mu\partial_\mu\boldsymbol{\psi} - m\overline{\boldsymbol{\psi}}\boldsymbol{\psi} + \lambda(\overline{\boldsymbol{\psi}}\boldsymbol{\psi})^2$, from which we may derive the nonlinear Dirac equation

$$\mathrm{i}\gamma^\mu\partial_\mu\boldsymbol{\psi} - m\boldsymbol{\psi} + 2\lambda(\overline{\boldsymbol{\psi}}\boldsymbol{\psi})\boldsymbol{\psi} = 0, \tag{1.1}$$

where the γ^μ matrices are defined by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

here σ^k with $k=1,2,3$, denote the Pauli matrices. The nonlinear self-coupling term $(\overline{\boldsymbol{\psi}}\boldsymbol{\psi})^2$ in the Lagrangian allows the existence of finite energy, localized solitary waves, or extended particle-like solutions, see e.g. [20]. Several authors have committed themselves to analytically investigating the Soler model, including stability [2,5–8,21], coupling with other fields [15,16], and some mathematical properties [13]. A review on the Soler model can be found in [14].

The current work is concerned with the numerical investigation of the interaction dynamics of Dirac solitary waves in the Soler model. Up to now, some reliable, higher-order accurate numerical methods have been constructed to solve the nonlinear Dirac equation (1.1). They include Crank-Nicholson type schemes [3,4], split-step spectral schemes [12], Legendre rational spectral methods [23], adaptive mesh method [22], and Runge-Kutta discontinuous Galerkin (RKDG) methods [17], etc. The interaction dynamics for the solitary wave solutions of (1.1) were numerically simulated in [4] by using a second-order accurate difference scheme. The authors saw there: charge and energy interchange except for some particular initial velocities of the solitary waves; inelastic interaction in binary collisions; and bound state production from binary collisions. Weakly inelastic interaction in ternary collisions is observed in [17]. The interaction dynamics in the binary and ternary collisions of two-humped solitary waves were first investigated in [18].

However, the experiments carried out in the literatures are all limited to the binary and ternary collisions of in-phase solitary waves of (1.1). In fact, the relative phase of those waves may vary with the interaction (see **Case B1** in Section 3), and their interaction generally depends on the initial phase shift. In this paper we will devote ourselves to further investigating the interaction dynamics in the binary and ternary collisions of the Dirac solitary waves with an initial phase shift by using the fourth-order accurate RKDG method [17] and the phase plane method, and report some interesting observations. The phase plane method [1] is based on the analysis of solitary wave trajectories on the phase plane. The RKDG methods for the Soler model adopt a discontinuous piecewise polynomial space for the approximate solutions and the test functions, and an explicit, high-order Runge-Kutta time discretization [9–11, 17]. Various experiments in [17, 18] have demonstrated that the fourth-order RKDG method is numerically stable without generating numerical oscillation within a very long time interval, has uniformly numerical convergence-rates, and preserves conservation of the energy and charge.

The paper is organized as follows. We introduce the $(1+1)$-dimensional space-time version of the NLD model (1.1) as well as its two solitary wave solutions in Section 2, and