Commun. Comput. Phys.
doi: 10.4208/cicp.2009.09.043

Vol. 7, No. 3, pp. 564-579
March 2010

Variational Formulation for Guided and Leaky Modes in Multilayer Dielectric Waveguides

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Received 4 March 2009; Accepted (in revised version) 6 August 2009

Communicated by Gang Bao

Available online 23 September 2009

Abstract. The guided and leaky modes of a planar dielectric waveguide are eigensolutions of a singular Sturm-Liouville problem. The modes are the roots of a characteristic function which can be found with several methods that have been introduced in the past. However, the evaluation of the characteristic function suffers from numerical instabilities, and hence it is often difficult to find all modes in a given range. Here a new variational formulation is introduced, which, after discretization, leads either to a quadratic or a quartic eigenvalue problem. The modes can be computed with standard software for polynomial eigenproblems. Numerical examples show that the method is numerically stable and guarantees a complete set of solutions.

AMS subject classifications: 65N25, 65F15, 35J20

Key words: Dielectric waveguide, polynomial eigenvalue problem, variational formulation, finite elements, singular Sturm-Liouville problem, continuation method.

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1 Introduction

Understanding the propagation properties of electromagnetic waves in layered dielectric media is essential for many applications in photonics. Multilayer dielectric structure can guide waves and are characterized by a refractive index that varies only in x-direction. Furthermore, in each layer the permittivity of each is a piecewise constant function, and a constant (with possibly different values) outside the finite interval $x \in [0,w]$. The two semi-infinite intervals are the cover and substrate and the finite interval is the stack of the waveguide. The magnetic permeability μ is constant.

The modes of such a structure have either transverse electric (TE) or transverse magnetic (TM) polarization. In the former case the electric field is of the form $\vec{E}(\vec{r},t) =$

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$\phi(x)\exp[i(\beta z-kt)]\cdot\vec{e}_y$ where ϕ satisfies the scalar equation

$$\phi''(x)+(k^2n^2(x)-\beta^2)\phi(x)=0, \quad x\in\mathbb{R}. \tag{1.1}$$

Here, $n(x)=\sqrt{\epsilon(x)\mu}$ is the refractive index.

In the TM case the magnetic field is of the form $\vec{H}(\vec{r},t)=\phi(x)\exp[i(\beta z-kt)]\cdot\vec{e}_y$ where ϕ satisfies the scalar equation

$$\left(\frac{1}{n^2(x)}\phi'(x)\right)'+\left(k^2-\frac{\beta^2}{n^2(x)}\right)\phi(x)=0, \quad x\in\mathbb{R}. \tag{1.2}$$

Equations (1.1) and (1.2) are singular Sturm-Liouville problems, where the propagation constant β is the unknown eigenvalue.

The spectrum consists of a discrete part, corresponding to guided modes, and a continuous part, corresponding to radiation modes. These modes form a complete set, that is, any function in $L_2(\mathbb{R})$ is a superposition of a finite number of guided modes and a continuum of radiation modes [11]. Using the framework of thin-film translation matrices it is easy to obtain a characteristic function whose real roots are propagation constants of the guided modes. With the same framework one can also derive the form of the modes in the continuous spectrum [5].

There is a third type of eigensolutions, known as leaky modes. These are unbounded solutions corresponding to complex roots of the characteristic function whose modes radiate energy away from the stack. Although leaky waves have infinite energy they are physically significant and have been verified experimentally in finite regions of the waveguide [19]. The leaky modes form a discrete set of expansion functions in the stack and can therefore represent field solutions in this region [12]. Leaky-wave analysis has the advantage that in the representation of a field the superposition integral of radiation modes is replaced by a discrete sum of leaky modes. In practice, only a few modes are necessary to obtain good approximations. This type of analysis has been applied in several photonics applications, see, e.g., [10] and the references cited therein. For more information on leaky modes in planar waveguides we refer to the recently published survey article [8].

Since waves can travel in two directions in the semi-infinite layers of the waveguide, the characteristic function has two branch cuts in the complex plane. To avoid the difficulties of the iterative root finder caused by the branch cuts, Smith et. al. [15] suggest a change of variables in which the characteristic function is analytic in the complex plane except for the origin. To find all roots in a specified region of the complex plane one can use a method by Delves and Lyness [6], which is based on the argument principle of complex analysis. Similar techniques for finding the propagation constants of dielectric waveguides are discussed in [1, 2, 4] and [16]. A related method to find the roots is by continuation from a closed to an open waveguide [9].

The core of these methods is the evaluation of a characteristic function. Unfortunately, the function exhibits the exponential scaling which can lead to numerical instabilities