Commun. Comput. Phys.
doi: 10.4208/cicp.180210.251110a

Vol. **10**, No. 4, pp. 979-1000
October 2011

Conservative Semi-Lagrangian Finite Difference WENO Formulations with Applications to the Vlasov Equation

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Received 18 February 2010; Accepted (in revised version) 25 November 2010

Available online 22 June 2011

Abstract. In this paper, we propose a new conservative semi-Lagrangian (SL) finite difference (FD) WENO scheme for linear advection equations, which can serve as a base scheme for the Vlasov equation by Strang splitting [4]. The reconstruction procedure in the proposed SL FD scheme is the same as the one used in the SL finite volume (FV) WENO scheme [3]. However, instead of inputting cell averages and approximate the integral form of the equation in a FV scheme, we input point values and approximate the differential form of equation in a FD spirit, yet retaining very high order (fifth order in our experiment) spatial accuracy. The advantage of using point values, rather than cell averages, is to avoid the second order spatial error, due to the shearing in velocity (v) and electrical field (E) over a cell when performing the Strang splitting to the Vlasov equation. As a result, the proposed scheme has very high spatial accuracy, compared with second order spatial accuracy for Strang split SL FV scheme for solving the Vlasov-Poisson (VP) system. We perform numerical experiments on linear advection, rigid body rotation problem; and on the Landau damping and two-stream instabilities by solving the VP system. For comparison, we also apply (1) the conservative SL FD WENO scheme, proposed in [22] for incompressible advection problem, (2) the conservative SL FD WENO scheme proposed in [21] and (3) the non-conservative version of the SL FD WENO scheme in [3] to the same test problems. The performances of different schemes are compared by the error table, solution resolution of sharp interface, and by tracking the conservation of physical norms, energies and entropies, which should be physically preserved.

AMS subject classifications: 65

Key words: Semi-Lagrangian methods, finite difference/finite volume scheme, conservative scheme, WENO reconstruction, Vlasov equation, Landau damping, two-stream instability.

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1 Introduction

In this paper, we propose a conservative semi-Lagrangian (SL) finite difference (FD) WENO scheme, by utilizing the same reconstruction procedure as in the SL finite volume (FV) WENO scheme in [3] for solving the 1-D advection equation

$$u_t+cu_x=0, \quad \text{where } c \text{ is a constant.} \tag{1.1}$$

This work is motivated by the kinetic plasma applications, where the Vlasov equation is often numerically solved by the following procedure. First, the Strang splitting is applied to decouple the high-dimensional nonlinear Vlasov equation into a sequence of linear advection equations, such as (1.1); then a SL scheme is apply to solve those decoupled 1-D equations. The SL approach for solving the Strang splitted Vlasov equation has been very popular in the plasma simulation community, see for example [3, 9, 12, 21, 27, 31], as the scheme for the splitted 1-D equation is usually simple, effective and free of CFL condition, which is a restriction in Eulerian approach. There are many variance in designing a SL scheme. Specifically, we characterize a SL scheme by the following three key components:

1. *A solution space.* The solution space can be point values, integrated mass (cell averages), or a piecewise polynomial function living on a fixed numerical grid, corresponding to the SL FD scheme [3,15,21], SL FV scheme [9,12] and the characteristic Galerkin method [5,17] respectively.

2. *Propagation.* In each of the time step evolution, information is propagated along characteristics. Usually, a high order interpolation or reconstruction procedure, which determines the spatial accuracy of the scheme, is applied to recover the information among discrete information on the solution space. In the literature, there are a variety of interpolation/reconstruction choices, such as the piecewise parabolic method (PPM) [7], positive and flux conservative method (PFC) [13], spline interpolation [8], cubic interpolation propagation (CIP) [28], ENO/WENO interpolation or reconstruction [3, 16, 21, 22, 26]. We refer to [9, 10, 29] for comparison of different reconstruction procedures.

3. *Projection.* Lastly, the evolved solution is projected back onto the solution space, updating the numerical solution at t^{n+1}.

It is known that the mass conservation is a very important property of a SL scheme. Failure to conserve the mass might lead to some instability of the scheme [15]. To conserve the mass, a scheme working with integrated mass in a FV spirit, seems more natural and straightforward [9]. On the other hand, we argue that it is advantageous to work with point values (FD scheme), rather than cell averages (FV scheme), due to the shearing of advection coefficients (v and E) over a cell, in the context of Strang splitting for Vlasov equation or other kinetic equations of similar kind. Due to above considerations, a conservative scheme that works with point values seems ideal [21, 22]. In this paper, we propose another approach of designing a conservative SL FD scheme by utilizing the