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Asymptotic Stability of an Eikonal Transformation Based ADI Method for the Paraxial Helmholtz Equation at High Wave Numbers

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Abstract. This paper concerns the numerical stability of an eikonal transformation based splitting method which is highly effective and efficient for the numerical solution of paraxial Helmholtz equation with a large wave number. Rigorous matrix analysis is conducted in investigations and the oscillation-free computational procedure is proven to be stable in an asymptotic sense. Simulated examples are given to illustrate the conclusion.

AMS subject classifications: 65M06, 65M12, 65F35, 15A12

Key words: Paraxial equation, highly oscillatory problems, eikonal splitting, asymptotic stability, matrix eigenvalues, spectral radius.

1 Introduction

Fast and accurate analysis of optical wave devices such as waveguides and couplers have been crucial to the development of light integrated systems [4, 13]. Core parts of such analysis often involve advanced computational procedures for investigating the wave propagation characteristics of the particular system. While the beam propagation method, which is based on fast Fourier transforms, has been popular in the study [7,8,17], different finite difference schemes are also employed in the research. To improve the accuracy of a numerical method used, a traditional approach is to increase the density of the grid or decrease the mesh step sizes utilized [4, 8, 9]. With a uniform mesh and step size, the cost for doing so may quickly become prohibitive if a high wave frequency is encountered. Nonuniform mesh structures and step sizes, on the other hand, may offer

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certain advantages in the situation [9,15]. However, they are often cumbersome to implement in industrial applications [3,13]. The issue of computational efficiency has become increasingly important in certain applications, in particular in the development of highly accurate, yet rapid numerical methods for solving paraxial, or parabolic, wave equations in order to separate inaccuracies inherent in numerical methods from inaccuracies due to paraxial wave approximations under modern laser configurations [12,13,16].

Consider a slowly varying envelope approximation of the light beam. A frequently used paraxial optical wave model is the Helmholtz equation,

$$2\mathbf{i}\kappa_0 l_0 E_z = E_{xx} + E_{yy} + \kappa_0^2 [l^2(x,y,z) - l_0^2] E, \qquad (x,y,z) \in \mathcal{D}, \tag{1.1}$$

where E is the electric field function of the light wave within a narrow cone, z is the beam propagation direction, x,y are transverse directions perpendicular to the light, $\mathbf{i}=\sqrt{-1}$, κ_0 is the wavenumber in free space, l_0 is the reference refractive index and $l(x,y,z)$ is the cross section index profile [4,7,9]. The differential equation provides solutions that describe the propagation of electromagnetic waves in the form of either paraboloidal waves or Gaussian beams. Most lasers emit beams that take the latter form [4,8]. The paraxial wave equation (1.1) can be viewed as a simplification of Maxwell's field equations [1,8,11,13]. Without loss of generality, we set $\mathcal{D} = \{a < x < b,\ c < y < d,\ z > z_0\}$.

Since the wave parameter $\kappa = \kappa_0 l_0$ is large in optical applications, the field function E is highly oscillatory. This may considerably impair our desire for a higher computational efficiency as well as accuracy in a traditional way, since mesh steps cannot be unrealistically small [6,14,18].

This motivates the latest search for eikonal transformation based numerical methods. The idea is straightforward. Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be sufficiently smooth real functions satisfying conditions

$$|\phi_z| \ll \kappa |\phi|, \qquad |\phi_{zz}| \ll \kappa^2 |\phi|, \qquad (x,y,z) \in \mathcal{D}. \tag{1.2}$$

We then look for the solution of (1.1) in the form of

$$E(x,y,z) = \phi(x,y,z) e^{\mathbf{i}\kappa\psi(x,y,z)}, \qquad (x,y,z) \in \mathcal{D}. \tag{1.3}$$

In fact, functions ϕ and ψ are closely related to the amplitude and ray, or eikonal, functions corresponding to the electric field E, respectively. The constraint (1.2) coincides with the basic feature of paraxial waves, that is,

$$\sin\theta \approx \theta, \qquad \tan\theta \approx \theta,$$

where θ is the angle between the beam vector and optical axis [1,8]. Transformations similar to (1.3) have also been used frequently in Wentzel-Kramers-Brillouin (WKB), or semiclassical, approximations in quantum physics.

The aim of this paper is not for a refinement of existing models, or a continue development of new schemes. Instead, we will focus at the numerical stability of the latest