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System Reduction Using an LQR-Inspired Version of Optimal Replacement Variables

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Abstract. Optimal Replacement Variables (ORV) is a method for approximating a large system of ODEs by one with fewer equations, while attempting to preserve the essential dynamics of a reduced set of variables of interest. An earlier version of ORV [1] had some issues, including limited accuracy and in some rare cases, instability. Here we present a new version of ORV, inspired by the linear quadratic regulator problem of control theory, which provides better accuracy, a guarantee of stability and is in some ways easier to use.

AMS subject classifications: 65M06, 65M30, 65M70

Key words: System reduction, optimal replacement variables, resolved variables, optimal prediction.

1 Introduction

The problem studied in this paper is that of approximating a large system of ordinary differential equations (ODEs) by one with a smaller number of equations.

This is desirable in many situations. For example, many partial differential equations, when discretized into a system of ODEs, require millions of degrees of freedom (DOF) to adequately approximate the dynamics. However, most of these DOF are usually of no interest. Examples of such PDEs include weather simulations and many simulations of fluid or aerodynamic flows. In the flow-around-an-aircraft example, an engineer would be mainly interested in bulk features such as the total lift and drag, or average vorticity as a function of time. A flow field detailed enough to actually resolve all of the dynamics would not be needed in many situations. Examples of linear systems of interest include

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the discretized versions of any linear PDE with complex geometry, such as heat flow in electronic devices or structural dynamics of skyscrapers or aircraft. Calculating solutions to these equations can require quite large amounts of computing resources, and a system reduction method such as the one studied here has the potential to reduce the resources required, or allow the fast solution of more complex problems.

The version of the problem studied here is a system of ODEs

$$z_t = Fz = \begin{pmatrix} F_0 & F_1 \\ F_2 & F_3 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad t \geq 0, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{1.1}$$

Only the case of linear systems of ODEs is considered here. Nonlinear ODEs are a topic for future work.

The state vector $z \in \mathbb{R}^{m+n}$ is divided into a set of resolved variables $x \in \mathbb{R}^m$ that one wants to observe or calculate, and a set of unresolved variables $y \in \mathbb{R}^n$ that one doesn't need to observe, but which the dynamics of the resolved variables x depend on. Furthermore, it may be that $m \ll n$ or even n infinite, in which case it will not be computationally feasible to solve the full system of equations.

An overview of different approaches to the problem can be found in Givon, Kupferman, Stuart [9]. One approach for system reduction has been developed by Chorin et al. [5–7] and Chertock, Gottlieb & Solomonoff [8], which has been called the T-System or Optimal Prediction.

Another approach, called Optimal Replacement Variables (ORV), see Solomonoff & Don [1] attempts to find a low-dimensional replacement for the unresolved variables such that the influence of the unresolved variables on the resolved ones is adequately approximated by the replacement variables. This is the approach which the present paper builds upon.

1.1 Other types of system reduction

There are several versions of the system reduction problem. We will call the one of this paper, the Optimal Prediction Problem (OPP). Others are similar in flavor, but differ enough in details that techniques used for one problem are of little use in another.

1.1.1 Control theory system reduction problem

The canonical control theory system is

$$x_t = Ax + Bu, \qquad y = Cx, \tag{1.2}$$

where u is a low-dimensional control input, y is a low-dimensional output and x is a high-dimensional internal state, and the goal is to construct a low-dimensional system that approximates the relation between the inputs and the outputs. Long-time behavior is emphasized and initial conditions tend to be ignored. This problem is quite well studied [15, 16].