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A Sub-Grid Structure Enhanced Discontinuous Galerkin Method for Multiscale Diffusion and Convection-Diffusion Problems

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Abstract. In this paper, we present an efficient computational methodology for diffusion and convection-diffusion problems in highly heterogeneous media as well as convection-dominated diffusion problem. It is well known that the numerical computation for these problems requires a significant amount of computer memory and time. Nevertheless, the solutions to these problems typically contain a coarse component, which is usually the quantity of interest and can be represented with a small number of degrees of freedom. There are many methods that aim at the computation of the coarse component without resolving the full details of the solution. Our proposed method falls into the framework of interior penalty discontinuous Galerkin method, which is proved to be an effective and accurate class of methods for numerical solutions of partial differential equations. A distinctive feature of our method is that the solution space contains two components, namely a coarse space that gives a polynomial approximation to the coarse component in the traditional way and a multiscale space which contains sub-grid structures of the solution and is essential to the computation of the coarse component. In addition, stability of the method is proved. The numerical results indicate that the method can accurately capture the coarse behavior of the solution for problems in highly heterogeneous media as well as boundary and internal layers for convection-dominated problems.

AMS subject classifications: 65M12, 65M60

Key words: Multiscale problem, sub-grid capturing, multiscale basis function, boundary layer, internal layer.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain in the two-dimensional space. We consider the following static convection-diffusion problem

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$$\mathcal{L}(u) \equiv \nabla\cdot(\vec{b}u - a\nabla u) = f, \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = g, \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where $\vec{b}$ is a given divergence-free vector field, f and g are given source and boundary functions. We also consider the corresponding time-dependent problem

$$\frac{\partial u}{\partial t} + \mathcal{L}(u) = f, \qquad \text{in } (0,T)\times\Omega, \tag{1.2a}$$

$$u = g, \qquad \text{on } (0,T)\times\partial\Omega, \tag{1.2b}$$

$$u(0,x) = u_0, \qquad \text{in } \Omega. \tag{1.2c}$$

In (1.1) and (1.2), we assume the ellipticity condition that $c_1 \geq a(x) \geq c_0 > 0$ for all $x \in \Omega$ and for some constants c_0 and c_1. Our aim in this work is the numerical approximation of (1.1) and (1.2) in the case when a and $\vec{b}$ are highly oscillatory or in the case when a is very small in some region that gives a convection-dominated diffusion problem. It is well-known that the solutions to these problems contain multiple scales, and the numerical computations require a very fine grid. Thus, a significant amount of computer memory and time are needed, and with the superior computing power nowadays, the computation of the solutions to these problems is still very challenging and sometimes even impossible. Nevertheless, the solutions to these problems typically contain a coarse component, which is usually the quantity of interest and can be represented by a small number of degrees of freedom. There are in literature many methods that aim at solving these problems on a coarse grid with great success. For example, see [5, 6, 12–15, 17, 21] for multiscale diffusion and wave problems, [16, 22] for multiscale convection-diffusion problems and [20] for two-phase flow problems.

The discontinuous Galerkin (DG) method is proved to be an effective and accurate class of tools for the numerical solutions of partial differential equations [1, 2, 4, 7–11]. The main idea is to use polynomial approximation on each cell without enforcing any continuity along cell interfaces. The success of these methods is achieved by using some sophisticated techniques to control the jumps. Due to the high efficiency and flexibility of DG methods, there are some advancement in using DG methods for the numerical approximation of problems with multiple scales. To the best of our knowledge, there are two existing classes of methods in literature. First of all, the discontinuous enrichment method has been proposed in [19] by Kalashnikova, Farhat and Tezaur. In this work, the solution space is discontinuous and contains two components, which is a polynomial space and a space spanned by the solution of local cell problem. One significant assumption is that the solutions of the local cell problems can be solved analytically. For problems with inhomogeneous media, the technique of frozen coefficient is applied. The formulation of the discrete problem is based on a DG framework, but the continuity is enforced by the method of Lagrange multiplier. The second class of method is the multiscale discontinuous Galerkin method proposed in [24] by Wang, Guzman and Shu. In this