Commun. Comput. Phys.
doi: 10.4208/cicp.101214.011015a

Vol. **20**, No. 1, pp. 86-135
July 2016

# Directly Simulation of Second Order Hyperbolic Systems in Second Order Form via the Regularization Concept

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Received 10 December 2014; Accepted (in revised version) 1 October 2015

**Abstract.** We present an efficient and robust method for stress wave propagation problems (second order hyperbolic systems) having discontinuities directly in their second order form. Due to the numerical dispersion around discontinuities and lack of the inherent dissipation in hyperbolic systems, proper simulation of such problems are challenging. The proposed idea is to denoise spurious oscillations by a post-processing stage from solutions obtained from higher-order grid-based methods (e.g., high-order collocation or finite-difference schemes). The denoising is done so that the solutions remain higher-order (here, second order) around discontinuities and are still free from spurious oscillations. For this purpose, improved Tikhonov regularization approach is advised. This means to let data themselves select proper denoised solutions (since there is no pre-assumptions about regularized results). The improved approach can directly be done on uniform or non-uniform sampled data in a way that the regularized results maintenance continuous derivatives up to some desired order. It is shown how to improve the smoothing method so that it remains conservative and has local estimating feature. To confirm effectiveness of the proposed approach, finally, some one and two dimensional examples will be provided. It will be shown how both the numerical (artificial) dispersion and dissipation can be controlled around discontinuous solutions and stochastic-like results.

**AMS subject classifications**: 35L51, 65N30

**Key words**: Second order hyperbolic systems, Tikhonov regularization, Numerical (artificial) dispersion.

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# 1 Introduction

Many high-order numerical schemes (e.g., finite difference, spectral, pseudo-spectral, and finite element methods) have been developed for resolving of elliptic and parabolic partial differential equations (PDEs). This is because, these systems have the inherent dissipation feature [16]. The hyperbolic systems (both the first and second order ones) do not show this inherent feature; even small errors can cause instability in these systems as spurious oscillations [16]. Oscillations in solutions containing discontinuities are known as the numerical dispersion. To remedy this problem, for first-order hyperbolic problems some effective approaches are developed. They include high resolution schemes and discontinuous Galerkin methods (by combining benefits of high-resolution methods with finite elements) [37,58]. In high resolution schemes, in smooth areas higher-order approximations (second, third, or higher ones) are used, and around discontinuities, first order approximation is utilized by a non-linear procedure. This strategy leads to a spurious-oscillation free results.

An effective way for solving second order hyperbolic PDEs is to rewrite them as a system of first order hyperbolic equations and then to simulate them with one of the above-mentioned schemes. This will lead to oscillation free results with small numerical dissipation [32]. The second order hyperbolic PDEs can also be solved in their original second order form. For this purpose, several approaches have been developed:

1. Using artificial dissipation without modifying the governing equations. For this case, dissipation is inherently added in evaluation procedures. In the time domain, the algorithmic dissipative time integration methods have been developed for removing spurious oscillations [20–22, 26]. For the spatial domain, inherent filtering concept is also developed in derivative estimations [69, 70],
2. Adding some artificial viscosity in the governing equations to stabilize the solution. This can be done by using local artificial viscosity around high-gradient zones in the spatial domain [12, 13]. Hughes [25] showed that this approach damps mainly the middle modes without affecting the lower and higher modes substantially. As artificial diffusion decreases accuracy of solutions considerably, methods using the artificial diffusion only in high frequency ranges were developed; such as, the spectral viscosity schemes [8, 9, 56]. This approach has been employed for both first and second order hyperbolic systems,
3. Filtering spurious oscillations from numerical solutions in the spatial domain by a post-processing stage [16, 35]. These schemes were successfully used in simulation of hyperbolic systems on uniform grid points [18, 59, 62], and non-uniform grids [66, 67]. It should be mentioned that many smoothing schemes working satisfactorily on uniform grids are not suitable for non-uniform ones: leading to unstable or unreliable results [39].

The concept of high-resolution treatment has recently been advised for handling second-order hyperbolic systems [3].