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Is Pollution Effect of Finite Difference Schemes Avoidable for Multi-Dimensional Helmholtz Equations with High Wave Numbers?

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Abstract. This paper presents an approach using the method of separation of variables applied to 2D Helmholtz equations in the Cartesian coordinate. The solution is then computed by a series solutions resulted from solving a sequence of 1D problems, in which the 1D solutions are computed using pollution free difference schemes. Moreover, non-polluted numerical integration formulae are constructed to handle the integration due to the forcing term in the inhomogeneous 1D problems. Consequently, the computed solution does not suffer the pollution effect. Another attractive feature of this approach is that a direct method can be effectively applied to solve the tridiagonal matrix resulted from numerical discretization of the 1D Helmholtz equation. The method has been tested to compute 2D Helmholtz solutions simulating electromagnetic scattering from an open large cavity and rectangular waveguide.

AMS subject classifications: 65N06, 65N15, 65N22

Key words: Helmholtz equation, separation of variables, high wave number, pollution free finite difference scheme, large cavity problem, rectangular waveguide.

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1 Introduction

In this paper, we consider the multi-dimensional Helmholtz equation

$$-\Delta u - k^2 u = f, \quad \text{in } \Omega, \tag{1.1}$$

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where Ω is a bounded domain, $k=\omega/c$ is the wave number with ω being the circular frequency, c and f represents the speed of sound and the forcing term, respectively.

The problem arises in many applications related to time harmonic wave propagations, and it has been used to model problems in acoustic, geophysics and electromagnetic wave scattering. There are many difficulties when simulating the Helmholtz equation with high wave numbers, since the solution becomes highly oscillatory when k is very large. The utmost challenge of solving the Helmholtz equation numerically is to eliminate or minimize the pollution error. Due to the pollution effect and in order to ensure the bounded of the relative error [30], conventional numerical algorithms require

$$k^{\gamma_1}(kh)^{\gamma_2}=C_1, \quad \gamma_1>0, \quad \gamma_2\geq 0, \tag{1.2}$$

where h denotes the mesh size and C_1 is a general positive constant. The parameters γ_1 and γ_2 depend on the choice of a specified numerical method. For instance, $\gamma_1=2$, $\gamma_2=2$ and $\gamma_1=1$, $\gamma_2=2$ for the standard finite difference scheme and the finite element method with P_1 element, respectively. Consequently, when the solution is computed by the finite difference and the finite element method, the mesh size $h=\mathcal{O}(1/k^2)$ and $h=\mathcal{O}(1/k^{3/2})$. Therefore, the impact of the pollution effect presents a serious challenge for multi-dimensional problems with high wave number k [5,19,30]. It should be pointed out that the resulting linear system is indefinite and very ill-conditioned, and it is not easy to develop a robust and accurate numerical solver for such systems [6,9,16,17,22,36,43, 59,60].

Many literatures have been reported devoting to eliminate or minimize the pollution effect, and the reader is referred to [10–12,14,15,24,25,27,34,37,41,42,44,45,48,51,53–56] for the finite difference method, and [1,3,5,19,20,30–33,40,49,57,61,62] for the finite element method. For other numerical techniques, the use of spectral approximation can be found in [46,47], and the boundary element method in [29,39]. In these literatures, three common approaches have been considered to overcome the pollution effect, namely, the higher-order method, the parameter method and the combined parameter-higher-order method. For the higher-order finite difference method, we refer to [12,25,27,37,41,44, 45,56]. In [45], Singer and Turkel proposed a fourth order finite difference scheme based on the Pade approximation. Fu [25] proved the error estimate for a compact fourth order method and claimed that the scheme is independent of the wave number. However, the theoretical analysis and numerical results (see [55]) revealed that the error of classical finite difference methods do depend on the wave number in multi-dimensional problems. Higher-order finite element method has been developed to minimize the pollution effect by increasing the order of the polynomial basic functions, and it is usually referred as the p-version FEM (see [1,30,32,40]). In a parameter method, the performance of the finite difference scheme is improved by choosing a suitable parameter [14,15,34,42,48,56]. Jo et al. [34] derived a rotated 9-points difference scheme by combining two second derivative operators for the 2D Helmholtz equation. Further investigation is reported in [48] by extending to the rotated 25-points finite difference scheme. Recently, Chen et al. adapted this idea and proposed a new strategy for choosing optimal parameter in [14] and con-