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An Efficient Spectral Petrov-Galerkin Method for Nonlinear Hamiltonian Systems

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Abstract. In this paper, an efficient spectral Petrov-Galerkin time-stepping method for solving nonlinear Hamiltonian systems is presented and studied. Conservation properties of the proposed method (including symplectic structure preserving and energy conservation) are discussed. Iterative algorithm on how to discretize the nonlinear term is introduced and the uniqueness, stability and convergence properties of the iterative algorithm are also established. Finally, numerical experiments are presented to verify the efficiency of our algorithm.

AMS subject classifications: 65M15, 65M70, 65N30

Key words: Nonlinear Hamiltonian system, spectral Petrov-Galerkin method, iterative algorithm, energy conservation, symplectic structure.

1 Introduction

Hamiltonian dynamical system was first introduced by Hamilton in 1824 as a general mathematical scheme for problems of geometrical optics. Since then, there is a growing interest of research and realization of the importance of Hamiltonian systems in many different areas such as classical mechanics, molecular dynamics, hydrodynamics, electrodynamics, plasma physics, relativity, astronomy, and so on [24, 25]. The numerical method or computational experiment is now considered as a very powerful technology and methodology for investigation and solving of the nonlinear Hamiltonian system since it is impossible to find exact solutions for most nonlinear Hamiltonian systems. A

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basic idea behind the design of numerical schemes is that they incorporate as many as possible properties of the original Hamiltonian dynamical system such as preservation of energy, preservation of momentum, symplecticity, reversibility in time, see, e.g. [2,22].

During the last decades, much work has been done in the field of designing momentum-conserving algorithms for Hamiltonian dynamical system, most important among which are symplectic algorithms and energy-momentum algorithms. Pioneering work on the symplectic geometry algorithm (that preserve the symplectic structure for Hamiltonian systems) can be found on Feng [10] and Ruth [28] and Channell [6]. Later, many different symplectic algorithms have been systematically developed and discussed, and many of them are Runge-Kutta methods. We refer to [5, 11–13, 21, 26, 29, 30] for an incomplete list of references in this field. Meanwhile, a number of algorithms have also been developed specifically for Hamiltonian systems to conserve the energy (see, e.g., [17, 18, 20, 34, 35]). For some highly oscillatory problems, energy preserving methods such as Störmer-Verlet-leapfrog methods, modulated Fourier expansion methods, have been developed to obtain long-time near-conservation of the total and oscillatory energies, see, e.g., [7, 8, 17, 18]. Given the importance of the Hamiltonian structure and energy-preserving properties of Hamiltonian systems, one might hope to find an algorithm which combines both of the desirable properties of the symplectic and energy-conservative algorithms. However, as is proved in [9, 38], there exists no energy preserving symplectic algorithm for general nonlinear Hamiltonian systems, and an energy-conservative algorithm would be at the expense of not being symplectic. Then we face a dilemma and have to choose between preserving energy and preserving symplectic structure. So far, it is not clear which algorithms should be preferred for a given application.

Due to the high order (spectral) accuracy, the spectral method has been widely used for numerical solutions of various differential equations, see, e.g., [3, 4, 14, 15, 31–33]. The spectral accuracy of the spectral method may provide us a new way to incorporate both properties of symplectic and energy-conservative algorithms for solving Hamiltonian systems in practice applications, with a negligible numerical error. Only recently, we present and study three kinds of efficient spectral methods (i.e., spectral Galerkin, Petrov-Galerkin, and collocation methods) for nonlinear Hamiltonian systems in [1]. We prove that the spectral Petrov-Galerkin method preserves the energy while both the spectral Gauss collocation and spectral Galerkin methods are energy conserving up to a spectral accuracy; and Petrov-Galerkin method preserves the symplectic structure up to a Gauss numerical quadrature error and the spectral Galerkin method preserves the symplectic structure up to a spectral accuracy error. In other words, in practice, all the three spectral methods are energy-preserving and symplectic structure up to a machine precision with a reasonable polynomial degree N.

The current work is the second in a series of study efficient algorithms for nonlinear Hamiltonian systems where the stability of the iterative algorithm and the application of the algorithm to various nonlinear Hamiltonian systems are under concern. Since the Gauss collocation method can be regarded as a special Runge-Kutta method, and