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How does Gauge Cooling Stabilize Complex Langevin?

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Abstract. We study the mechanism of the gauge cooling technique to stabilize the complex Langevin method in the one-dimensional periodic setting. In this case, we find the exact solutions for the gauge transform which minimizes the Frobenius norm of link variables. Thereby, we derive the underlying stochastic differential equations by continuing the numerical method with gauge cooling, and thus provide a number of insights on the effects of gauge cooling. A specific case study is carried out for the Polyakov loop model in $SU(2)$ theory, in which we show that the gauge cooling may help form a localized distribution to guarantee there is no excursion too far away from the real axis.

AMS subject classifications: 65C05, 65C30
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1 Introduction

In quantum chromodynamics (QCD), the renormalization of the coupling constant depends on the energy scale. As the energy scale increases, the coupling constant decays to zero. Therefore the perturbative theory works well for high-energy scattering. However, when studying QCD at small momenta or energies (less than 1GeV), the coupling constant is comparable to 1 and the perturbative theory is no longer accurate [16]. In this case, one of the important methods is the path integral formulation, in which people usually employ the lattice gauge theory to perform calculations. In lattice QCD, the degrees of freedom for both gluons and quarks are discretized on a four-dimensional lattice, whose grid points are

$$\boldsymbol{x} = (i,j,k,t)a, \qquad i,j,k,t \in \mathbb{Z},$$

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where a is the size of the lattice. Gluons on the lattice are represented by link variables between lattice points, which are matrices $U_\mu(\boldsymbol{x}) \in SU(3)$, denoting the link between lattice points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{e}_\mu$. Since the degrees of freedom for quarks can usually be integrated out explicitly, the final form of the path integral is given by the following partition function:

$$Z = \int [\mathrm{d}U] \det M(\{U\}) \mathrm{e}^{-S(\{U\})}, \tag{1.1}$$

where $\int [\mathrm{d}U]$ stands for the integral with respect to all link variables $U_\mu(\boldsymbol{x})$ defined on the Haar measure of $SU(3)$, and $\{U\}$ represents the collection of all link variables. In the integrand, the matrix $M(\{U\})$ is the fermion Green's function, and $S(\{U\})$ is the Euclidean action for the gluons. Thus, given an observable $O(\{U\})$, its expected value can be calculated by

$$\langle O \rangle = \frac{1}{Z} \int [\mathrm{d}U] O(\{U\}) \det M(\{U\}) \mathrm{e}^{-S(\{U\})}.$$

Monte Carlo methods such as the Metropolis algorithm and the Langevin algorithm can be applied to evaluate this integral.

When we consider the system with quark chemical potential, the term $\det M(\{U\})$ may be non-positive, and thus (1.1) does not allow for a probabilistic interpretation [11]. In such a circumstance, the "reweighting" technique is required to carry out the Monte Carlo simulation. Such a method introduces another partition function

$$Z_0 = \int [\mathrm{d}U] \det M_0(\{U\}) \mathrm{e}^{-S(\{U\})}$$

and rewrite Z as

$$Z = Z_0 \left\langle \frac{\det M(\{U\})}{\det M_0(\{U\})} \right\rangle_0,$$

where $\langle \cdot \rangle_0$ denotes the expectation of $\cdot$ based on the partition function Z_0. However, due to the rapid change of phase in $\det M(\{U\}) / \det M_0(\{U\})$, significant numerical sign problem may appear, causing large deviation in the numerical integration [10].

To relax the sign problem, numerical methods such as Lefschetz thimble method [9] and complex Langevin method (CLM) [19–21, 26] have been introduced. This paper focuses on the CLM, which can be considered as a straightforward complexification of the real Langevin method. While the complex Langevin method effectively relaxes the sign problem for some problems, the behavior of this method seems quite unpredictable. Although in a lot of cases, the method produces correct integral values, sometimes it provides incorrect integral values, or even generates divergent dynamics [5]. A number of efforts have been made to figure out the reason of failure and find a theory for its correct convergence [4, 12, 13, 23, 27]. Although a complete theory has not been found, the problem has been much better understood.