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A Multi-Scale DNN Algorithm for Nonlinear Elliptic Equations with Multiple Scales

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Abstract. Algorithms based on deep neural networks (DNNs) have attracted increasing attention from the scientific computing community. DNN based algorithms are easy to implement, natural for nonlinear problems, and have shown great potential to overcome the curse of dimensionality. In this work, we utilize the multi-scale DNN-based algorithm (MscaleDNN) proposed by Liu, Cai and Xu (2020) to solve multi-scale elliptic problems with possible nonlinearity, for example, the p-Laplacian problem. We improve the MscaleDNN algorithm by a smooth and localized activation function. Several numerical examples of multi-scale elliptic problems with separable or non-separable scales in low-dimensional and high-dimensional Euclidean spaces are used to demonstrate the effectiveness and accuracy of the MscaleDNN numerical scheme.

AMS subject classifications: 65N30, 35J66, 41A46, 68T07

Key words: Multi-scale elliptic problem, p-Laplacian equation, deep neural network (DNN), variational formulation, activation function.

1 Introduction

In this paper, we will introduce a DNN based algorithm for the following elliptic equation with multiple scales and possible nonlinearity

$$\begin{cases} -\mathrm{div}\Big(a(\boldsymbol{x},\nabla u(\boldsymbol{x}))\Big)=f(\boldsymbol{x}), & \text{in } \Omega, \\ u(\boldsymbol{x})=g(\boldsymbol{x}), & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal (polyhedral) domain (open, bounded and connected), $a(\boldsymbol{x}, \nabla u(\boldsymbol{x})) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is the flux function, and $f : \Omega \to \mathbb{R}$ is the source term.

Deep neural networks (DNNs) has not only achieved great successes in computer vision, natural language processing and other machine learning tasks [19, 28], but also captured great attention in the scientific computing community due to its universal approximating power, especially in high dimensional spaces [46]. It has found applications in the context of numerical solution of ordinary/partial differential equations, integral-differential equations and dynamical systems [16, 20, 26, 36, 41, 47].

Recent theoretical studies on DNNs have shed some light on the design of DNN-based algorithms for scientific computing tasks, in particular, for multi-scale problems. For example, the frequency principle (F-Principle) [15, 37, 44, 45], shows that, DNNs often fit target functions from low frequency components to high frequency ones, as opposed to the behavior of many conventional iterative numerical schemes (e.g., Gauss-Seidel method), which exhibit faster convergence for higher frequencies. To improve the convergence for high-frequency or multi-scale problems, a series of algorithms are developed to accelerate the learning of high-frequency components based on F-Principle [5, 6, 27, 30]. In particular, a multi-scale DNN algorithm(MscaleDNN) has achieved favourable performance boost for high-frequency problems [30]. The idea of the MscaleDNN to convert high-frequency contents into low-frequency ones as follows. The Fourier space is partitioned with respect to the radial direction. Since scaling input can shift the frequency distribution along the radial direction, a scaling down operation is used to scale the high-frequency components to low-frequency ones. Such radial scaling is independent of dimensionality, hence MscaleDNN is applicable for high-dimensional problems. Also, borrowing the multi-resolution concept of wavelet approximation theory using compact scaling and wavelet functions, an localized activation function (i.e., sReLU) was designed in previous work [30], which is a product of two ReLU functions. By setting multiple scalings in a MscaleDNN, numerical results in previous study [30] show that MscaleDNN is effective for linear elliptic partial differential equations with high frequencies.

We focus our exposition on the numerical method, and therefore restrict the flux function in (1.1) to the following Leray-Lions form [13] since it admits a natural variational form. Namely, for $(\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega \times \mathbb{R}^d$, $a(\boldsymbol{x}, \boldsymbol{\xi}) = \kappa(\boldsymbol{x}) \boldsymbol{\phi}'(|\boldsymbol{\xi}|) \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$, where $\boldsymbol{\phi} \in C^2$ is the so-called $N-$function (an extension for the convex function with $\boldsymbol{\phi}'(0) = 0$, see [13] for the precise definition). For p-Laplacian problem, $\boldsymbol{\phi}(t) = \frac{1}{p} t^p$, and when $p = 2$ then $a(\boldsymbol{x}, \boldsymbol{\xi}) = \kappa(\boldsymbol{x}) \boldsymbol{\xi}$ becomes linear. $\kappa(\boldsymbol{x}) \in L^\infty(\Omega)$ is symmetric, uniformly elliptic on Ω, and may contain (non-separable) multiple scales. More general nonlinear flux will be treated in future work. With the above setup, the elliptic problem (1.1) is monotone and coercive, therefore it admits a unique solution. Those models have applications in many areas such as heterogeneous (nonlinear) materials [18], non-Newtonian fluids, surgical simulation, image processing, machine learning [40], *etc*.

In the last decades, much effort has been made for the numerical solution of the (1.1). In particular, for p-Laplacian equation with $\kappa(\boldsymbol{x}) = 1$, Some degrees of effectiveness can