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Adaptive Order WENO Reconstructions for the Semi-Lagrangian Finite Difference Scheme for Advection Problem

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Abstract. We present a new conservative semi-Lagrangian finite difference weighted essentially non-oscillatory scheme with adaptive order. This is an extension of the conservative semi-Lagrangian (SL) finite difference WENO scheme in [Qiu and Shu, JCP, 230 (4) (2011), pp. 863-889], in which linear weights in SL WENO framework were shown to not exist for variable coefficient problems. Hence, the order of accuracy is not optimal from reconstruction stencils. In this paper, we incorporate a recent WENO adaptive order (AO) technique [Balsara et al., JCP, 326 (2016), pp. 780-804] to the SL WENO framework. The new scheme can achieve an optimal high order of accuracy, while maintaining the properties of mass conservation and non-oscillatory capture of solutions from the original SL WENO. The positivity-preserving limiter is further applied to ensure the positivity of solutions. Finally, the scheme is applied to high dimensional problems by a fourth-order dimensional splitting. We demonstrate the effectiveness of the new scheme by extensive numerical tests on linear advection equations, the Vlasov-Poisson system, the guiding center Vlasov model as well as the incompressible Euler equations.

AMS subject classifications: 65

Key words: Semi-Lagrangian, weighted essentially nonoscillatory, WENO adaptive order reconstruction, finite difference, mass conservation, Vlasov-Poisson, incompressible Euler.

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1 Introduction

In this paper, we propose a conservative semi-Lagrangian (SL) finite difference (FD) weighted essentially non-oscillatory adaptive order (WENO-AO) scheme for the advec-

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tion equation in the form of

$$u_t+\nabla_{\mathbf{x}}\cdot(\mathbf{P}(\mathbf{x},t)u)=0, \quad (\mathbf{x},t)\in\mathbb{R}^d\times[0,T], \tag{1.1}$$

with applications to kinetic equations such as the Vlasov-Poisson (VP) system, the guiding center Vlasov model as well as incompressible Euler equations.

For hyperbolic type advection equations, the Eulerian approach, e.g. Runge-Kutta discontinuous Galerkin (DG) and WENO methods [11], is well-known to be computationally effective with high order accuracy for smooth problems, preservation of local mass conservation, and non-oscillatory capture of shocks. One bottleneck of the Eulerian approach is the stringent Courant-Friedrichs-Lewy (CFL) condition for explicit time stepping. To overcome the CFL condition, the SL approach have been developed. In an SL framework, the solutions is updated by tracking information along characteristics; as a result, the scheme can remain stable for a much larger time stepping size than its Eulerian counterpart. SL algorithms have been applied for a wide range of application fields from climate modeling [27] to kinetic description of plasmas [16,33].

Preservation of mass conservation has been a top priority when designing schemes for hyperbolic conservation laws. For semi-Lagrangian schemes, the conservation property can be assured conveniently in a finite volume [13,20,23] or DG [8,14,24,30,31] frameworks. However, it is nontrivial to design a *conservative* SL scheme in the FD framework. In [28], a class of high order conservative SL WENO schemes is proposed for advection equations, yet it was shown in the same work there do not exist linear weights in the SL WENO scheme. As a result, the SL WENO scheme cannot achieve the optimal high order accuracy from its reconstruction stencils. To overcome this, we propose to apply other types of WENO methods that use a set of artificial weights and retains high order accuracy. Examples of such WENO reconstructions include CWENO [25, 26], WENO-ZQ [41], WENO-AO [4], targeted ENO scheme [17,18], hybrid WENO [1,35,40] and their extensions. Especially, there have been systematic studies on WENO-AO [2, 3, 22]. As in classic WENO-Z [6], smoothness indicators in WENO-ZQ and WENO-AO are properly defined for high order accuracy at critical points. Following the same spirit as the WENO-AO reconstruction [4], we propose new conservative SL FD WENO-AO schemes. The introduction of the WENO-AO procedure enables us to derive explicit formulas of the compact flux reconstruction with the full order in the SL framework. As mentioned in [28], designing the compact flux functions are critical for stability consideration, and designing WENO scheme for the composition of two reconstruction procedures is highly challenging. The proposed schemes in this paper can achieve the formal $(2r-1)$-th order accuracy when using a stencil of width $(2r-1)$. For high dimensional problems, we apply a fourth order dimensional splitting [37,38]. For problems with positivity physical variable such as the nonlinear VP system, we further apply a positivity preserving (PP) limiter in [36] that maintains high order accuracy of the schemes.

The organization of the paper is as follows: in Section 2, we present the detailed implementation of the SL FD WENO-AO schemes; in Section 3, we demonstrate the effectiveness of the proposed schemes by numerically testing one-dimensional (1D) and