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A Hybrid WENO Scheme for Steady Euler Equations in Curved Geometries on Cartesian Grids

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Abstract. For steady Euler equations in complex boundary domains, high-order shock-capturing schemes usually suffer not only from the difficulty of steady-state convergence but also from the problem of dealing with physical boundaries on Cartesian grids to achieve uniform high-order accuracy. In this paper, we utilize a fifth-order finite difference hybrid WENO scheme to simulate steady Euler equations, and the same fifth-order WENO extrapolation methods are developed to handle the curved boundary. The values of the ghost points outside the physical boundary can be obtained by applying WENO extrapolation near the boundary, involving normal derivatives acquired by the simplified inverse Lax-Wendroff procedure. Both equivalent expressions involving curvature and numerical differentiation are utilized to transform the tangential derivatives along the curved solid wall boundary. This hybrid WENO scheme is robust for steady-state convergence and maintains high-order accuracy in the smooth region even with the solid wall boundary condition. Besides, the essentially non-oscillation property is achieved. The numerical spectral analysis also shows that this hybrid WENO scheme has low dispersion and dissipation errors. Numerical examples are presented to validate the high-order accuracy and robust performance of the hybrid scheme for steady Euler equations in curved domains with Cartesian grids.

AMS subject classifications: 65N06, 35L65, 76M20

Key words: Euler equations, steady-state convergence, curved boundary, Cartesian grids, WENO extrapolation, hybrid scheme.

1 Introduction

The steady-state Euler equations play an important role in the shape optimal design of aircraft and vehicles. There are three significant difficulties in satisfactorily solving these equations. First, the detailed features of strong discontinuities are hard to be resolved by

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the numerical scheme. Second, the residual is difficult to converge to an ideal level, close to machine zero. Third, the complex physical boundary should be treated carefully to avoid large errors introduced near the border.

In order to resolve the strong discontinuities with high resolution, the high-order WENO (weighted essentially non-oscillatory) schemes are popular choices to discretize the spatial derivative. There are many WENO schemes developed to solve the unsteady Euler equations, such as the WENO-JS [23, 40], WENO-Z [3, 5] schemes, etc. These schemes not only can obtain the optimal order of accuracy in the smooth region, but also can achieve the essentially non-oscillatory property. To improve the efficiency of WENO schemes, researchers often adopt hybrid approaches [29–31, 64, 65, 75, 77]. Usually, the computational domain is divided into two parts: the smooth region and the non-smooth region. In the smooth region, the high-order linear scheme is used. In the non-smooth region, a certain WENO scheme is used. When the smooth region takes up most of the computational domain and the linear scheme is less costly, the hybrid approach can save significant computational time. Meanwhile, the effect of the hybrid scheme is heavily based on the proposed smoothness detector. Many types of shock detectors have appeared and people can refer to these literature [10,12,16,17,20,28,52,58,59,66,68,71,78]. If readers are interested in the history of WENO schemes, we refer to [32, 35, 55, 76] for more details.

Many studies have shown that the classical WENO and hybrid WENO schemes are efficient choices for the simulation of unsteady flows; however, when applied to the solution of steady-state problems with strong discontinuities, one is confronted with the second difficulty, i.e. the residual of the numerical solution usually stops at the level of the truncation error rather than settling down to the machine zero. To improve steady-state convergence, Serna and Marquina reconstructed the numerical flux by a new kind of limiter in [53]. Zhang and Shu [73, 74] pointed out that the appearance of slight post-shock oscillations has a tremendous impact on steady-state convergence, which is the key reason for the defective steady-state convergence phenomenon. After that, Zhang et al. proposed the upwind-biased interpolation technique [72,74] to improve steady-state convergence. However, the convergence phenomenon still seems unsatisfactory when shocks pass the physical boundary. In [7, 8], the authors solved steady-state hyperbolic conservation laws using fast sweeping methods with Lax-Friedrichs numerical fluxes and improved steady-state convergence by designing novel multigrid fast sweeping methods. Engquist et al. [13, 14] further pointed out that fast sweeping methods efficiently solve steady-state conservation laws. Hao et al. [19] utilized the homotopy method to solve the nonlinear system obtained by WENO discretizations. The nonlinear system should be treated carefully since its solutions are not unique. Liu et al. [38] developed an adaptive Runge-Kutta discontinuous Galerkin method to solve both unsteady and steady Euler equations on Cartesian grids, and the robustness of steady-state convergence still needs to be enhanced. Hu et al. [21, 22] used non-oscillatory k-exact reconstruction to solve steady-state Euler equations and enhanced the approximation accuracy of curved boundaries by nonuniform rational B-splines [45, 46]. Chen [6] and Wu et al. [69, 74] pro-