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On Single Distribution Lattice Boltzmann Schemes for the Approximation of Navier Stokes Equations

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Abstract. In this contribution we study the formal ability of a multi-resolution-times lattice Boltzmann scheme to approximate isothermal and thermal compressible Navier Stokes equations with a single particle distribution. More precisely, we consider a total of 12 classical square lattice Boltzmann schemes with prescribed sets of conserved and nonconserved moments. The question is to determine the algebraic expressions of the equilibrium functions for the nonconserved moments and the relaxation parameters associated to each scheme. We compare the fluid equations and the result of the Taylor expansion method at second order accuracy for bidimensional examples with a maximum of 17 velocities and three-dimensional schemes with at most 33 velocities. In some cases, it is not possible to fit exactly the physical model. For several examples, we adjust the Navier Stokes equations and propose nontrivial expressions for the equilibria.

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1 Introduction

The study of fluid mechanics is a natural problem set by the pioneers of the lattice Boltzmann schemes in their modern form (see [29,31,39,51] and many others). An underlying

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lattice Boltzmann equation is discretised on a cartesian grid with a finite set of velocities chosen in such a way that during one time step, an exact transport is done between two vertices of the mesh. A lattice Boltzmann scheme is composed with two steps: a nonlinear local relaxation step, followed by a linear advection scheme coupling a given vertex with a given family of neighbours. The relaxation step follows in general the approximation introduced by Bhatnagar, Gross and Krook [3].

From a completely defined lattice Boltzmann model (elementary velocity set, equilibrium and relaxation rates) it is possible to determine the macroscopic behaviour of the model through equivalent partial differential equations. One tries to match these equivalent equations to those governing a given physical situation. Instead of matching equivalent partial differential equations and physical partial differential equations, term by term, it is common place to compute the "hydrodynamic" modes of the two approaches and try to match them.

This process allows the design of new simulation techniques for an important number of physical phenomena as isothermal flows, compressible flows with heat transfer, non-ideal fluids, multiphase and multi-component flows, microscale gas flows, soft-matter flows,$\cdots$ up to quantum mechanics. The lattice Boltzmann method is inspired from a mesoscopic Boltzmann model but is not able in general to solve the Boltzmann equation or associated kinetic models. It is admitted that conservative macroscopic models can be approximated with the lattice Boltzmann schemes. For the present status of the method and the various applications, we refer *e.g.* to the books of Guo and Shu [23] or Krüger *et al.* [35].

The simulation of incompressible flows, isothermal flows or thermal flows with moderate compressible effects is very classical and we refer among others to the contributions [6, 18, 22, 26, 33, 39, 41, 47, 59] and to some operational softwares like OpenLB [30], Palabos [43] Powerflow [4], LaBS-ProLB [64] or pylbm [21]. In his prospective article [60], Succi points the fact that the simulation of compressible flows including the presence of two thermodynamic variables and eventually high Mach numbers is one of the main open questions related to lattice Boltzmann schemes.

Following an idea initially proposed in [34,56], a popular approach consists in adding a second particle distribution to treat the conservation of energy. Following this framework, Guo *et al.* [24] use two particle distributions on the standard D2Q9 lattice to simulate compressible thermal flows at low Mach number, Nie *et al.* [46] propose a double discrete distribution for thermal lattice Boltzmann model using a three-dimensional scheme employing 121 velocities, Latt *et al.* [44] use a double-distribution-function based on the D3Q39 scheme for the simulation of polyatomic gases in the supersonic regime, Frapolli *et al.* [17] present a huge variety of three-dimensional lattice Boltzmann models with two particle distributions to simulate compressible flows with the entropic lattice Boltzmann method. We proposed in [14] to recover the full Navier Stokes equations in one space dimension using two lattice Boltzmann schemes and treating an entropy equation with a second particle distribution.

The approximation of thermal Navier Stokes equations with lattice Boltzmann