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Energy-Stable Parametric Finite Element Methods for the Generalized Willmore Flow with Axisymmetric Geometry: Closed Surfaces

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Abstract. In this paper, we extend the work [2] to establish an energy-stable parametric finite element approximation for the generalized axisymmetric Willmore flow with closed surfaces. A crucial aspect of our approach is the introduction of two novel geometric identities involving the weighted normal velocity, $\vec{x}\cdot\vec{e}_1\,(\vec{x}_t\cdot\vec{\nu})\vec{\nu}$, and the curvature variable, $G_{\mathcal{S}}=\varkappa_{\mathcal{S}}-\overline{\varkappa}$, where $\varkappa_{\mathcal{S}}$ represents the mean curvature and $\overline{\varkappa}$ denotes the spontaneous curvature. We theoretically prove that the numerical method preserves the stability of the original energy. Several numerical tests are provided to demonstrate the energy stability, as well as the accuracy and efficiency of our developed numerical scheme.

AMS subject classifications: 65M60, 65M12, 53C44, 35K55

Key words: Generalized Willmore flow, parametric finite element method, energy-stable, geometric identity.

1 Introduction

Functionals incorporating the principal curvatures of two-dimensional surfaces are highly significant in mechanics, geometry, imaging, and biology, with their applications in plate and shell theories tracing back to the work of [14, 18, 22]. Typical membrane energies are closely related to the curvature of the membrane. In the simplest models, the Willmore functional, defined as the integral of the squared mean curvature, is often regarded as an appropriate energy, see [23]. Variational problems are of particular interest, with one notable result being the famous Willmore conjecture. This conjecture posits that the surface

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minimizing the Willmore energy among genus-1 surfaces is the Clifford torus, a problem that was recently resolved by [20]. In the field of imaging, the Willmore functional has been used to address boundary value problems related to image inpainting and surface restoration [7,10]. In biological membrane and vesicle theory, the role of surface area and volume constraints, along with more general curvature functionals, is significant. In [8], the shape of human red blood cells was explained by employing a curvature functional along with constraints on volume and surface area, with the membrane being represented as a two-dimensional surface.

In this work, we consider a type of generalized Willmore energy of the surface $\mathcal{S}(t)$ [15], defined by

$$\frac{\alpha}{2}\int_{\mathcal{S}}[\kappa_m-\overline{\varkappa}]^2 d\mathcal{H}^2+\alpha_G\int_{\mathcal{S}}\kappa_g d\mathcal{H}^2, \tag{1.1}$$

where κ_m is the mean curvature, $\overline{\varkappa}\in\mathbb{R}$ is a given constant known as the spontaneous curvature that reflects a possible asymmetry in the membrane, κ_g is the Gaussian curvature, $d\mathcal{H}^2$ denotes integration with respect to two-dimensional surface measure, and α and α_G represent the bending rigidities. In biological applications, membranes at equilibrium minimize the energy functional under the constraints of volume and surface area on the membrane surface $\mathcal{S}$.

As an initial attempt, we in this work consider the generalized Willmore flow under a closed surface. In this case, using the Gauss-Bonnet theorem, we have the following topological invariant

$$\int_{\mathcal{S}}\kappa_g d\mathcal{H}^2=2\pi m(\mathcal{S}), \tag{1.2}$$

with $m(\mathcal{S})$ representing the Euler characteristic of $\mathcal{S}$. Then, the following generalized axisymmetric Willmore flow, i.e., the L^2-gradient flow for the energy (1.1) [3], is given by

$$\begin{aligned}\frac{1}{\alpha}\mathcal{V}_{\mathcal{S}}&=-\Delta_{\mathcal{S}}\kappa_m-(\kappa_m-\overline{\varkappa})|\nabla_{\mathcal{S}}\vec{n}_{\mathcal{S}}|^2+\frac{1}{2}(\kappa_m-\overline{\varkappa})^2\kappa_m\\&=-\Delta_{\mathcal{S}}\kappa_m+2(\kappa_m-\overline{\varkappa})\kappa_g-\frac{1}{2}\left(\kappa_m^2-\overline{\varkappa}^2\right)\kappa_m\quad\text{on }\mathcal{S}(t),\end{aligned} \tag{1.3}$$

where $\mathcal{V}_{\mathcal{S}}$ is the normal velocity of an evolving surface $\mathcal{S}(t)$ for $t\in[0,T]$, $\Delta_{\mathcal{S}}:=\nabla_{\mathcal{S}}\cdot\nabla_{\mathcal{S}}$ represents the surface Laplace-Beltrami operator with $\nabla_{\mathcal{S}}$ being the surface gradient of $\mathcal{S}(t)$, and $\nabla_{\mathcal{S}}\vec{n}_{\mathcal{S}}$ denotes the Weingarten map.

In this work, our primary focus is on developing an energy-stable algorithm for the generalized Willmore flow (1.3) with axisymmetric geometry. When the evolving three-dimensional surface exhibits rotational symmetry, we can reduce the geometric flow problem to a simpler one-dimensional form. This approach significantly minimizes computational complexity, eliminates the need for intricate mesh controls by dealing with a one-dimensional generating curve, and preserves the axisymmetric property throughout