Commun. Comput. Phys.
doi: 10.4208/cicp.240813.170614a

Vol. **17**, No. 1, pp. 47-78
January 2015

A Differential Algebraic Method for the Solution of the Poisson Equation for Charged Particle Beams

B. Erdelyi[1,*], E. Nissen[2] and S. Manikonda[3]

[1] *Department of Physics, Northern Illinois University, DeKalb, IL, 60115, USA.*
[2] *European Organization for Nuclear Research, CH-1211 Geneva 23, Switzerland.*
[3] *Advanced Magnet Lab, Palm Bay, FL, 32905, USA.*

Received 24 August 2013; Accepted (in revised version) 17 June 2014

Communicated by Chi-Wang Shu

Abstract. The design optimization and analysis of charged particle beam systems employing intense beams requires a robust and accurate Poisson solver. This paper presents a new type of Poisson solver which allows the effects of space charge to be elegantly included into the system dynamics. This is done by casting the charge distribution function into a series of basis functions, which are then integrated with an appropriate Green's function to find a Taylor series of the potential at a given point within the desired distribution region. In order to avoid singularities, a Duffy transformation is applied, which allows singularity-free integration and maximized convergence region when performed with the help of Differential Algebraic methods. The method is shown to perform well on the examples studied. Practical implementation choices and some of their limitations are also explored.

PACS: 29.27.-a, 41.75.-i, 41.20.Cv, 05.10.-a

Key words: Charged particle beams, mean field limit, Poisson equation, orthogonal polynomials, differential algebra.

---

1 Introduction

Beams are ensembles of particles in directed motion. Charged particle beams underlie the science of particle accelerators with a variety of applications that span all science from physics to biology [1]. As accelerator technology improves, the usefulness of high intensity beams continues to increase, as illustrated by several existing and planned machines at the intensity frontier [2]. In order to properly design new accelerators and analyze

*Corresponding author. *Email addresses:* berdelyi@niu.edu (B. Erdelyi), nissen@jlab.org (E. Nissen), smanikonda@magnetlab.com (S. Manikonda)

existing machines to improve performance, a fast and accurate method of modeling and simulating intense charged particle beams is required. Quantitatively, the phenomena arising from the average self-fields of beams, i.e. the electrostatic mean field limit, is called space charge. The beam dynamics that includes both externally applied fields as well as the space charge fields is paramount to understand in order to design better future machines and improve operational performance of existing machines. In order to determine these fields, we must solve Poisson's equation. With open boundary conditions it is,

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \tag{1.1}$$

where Φ is the scalar potential to vanish at infinity, and which is due to a smooth charge density distribution ρ.

There are several methods to perform these simulations, most popular being the particle in cell method [3]. Particle in cell methods involve solving the Poisson equation on a grid, usually aided by (fast) Fourier transform methods. It uses various charge deposition and force interpolation algorithms. Another alternative is to evolve in time the one-particle phase space distribution of the beams using the method of characteristics, or by the direct solution of the corresponding Vlasov equation in the six dimensional phase space. In any case, a requirement for accurate and efficient examination of the effects of space charge is a powerful Poisson solver. Examples of methods for calculating the potential are the Fourier transform methods [4] and wavelet methods [5]. Other methods involve fitting the charge distribution to a series of functions which are then used to determine the potential, one example of which is the Green's function method [6]. In fact, there is a long history in beam and accelerator physics related to Poisson solvers, including their comparison. A particularly illuminating summary is [7] and references therein. Here we detail our novel Poisson solver, for which preliminary results and applications can be found in [8–11].

Before we go further detailing our Poisson solver, it is important to mention another method that is extensively developed for single-particle beam dynamics: the transfer map method [12, 13]. Transfer maps give the functional dependence between some final and initial conditions and contain all information about the system dynamics. In this context, normal form methods applied to transfer maps are powerful methods that allow a comprehensive analysis of low intensity beam dynamics [12, 13]. In order to extend the method to include space charge dynamics self-consistently, clearly the beam self-potential needs to be specified. Since the transfer map method requires an analytic polynomial representation, the question arises how to obtain such a representation from a particle distribution. This paper shows that with the help of Differential Algebraic (DA) methods [13], and some innovative algorithmic developments presented here, it is possible to obtain high order Taylor expansions of the beam self-potential around a point that is situated inside the charge distribution. For the purpose of this paper, we will restrict ourselves to open boundary conditions.