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Weak-Strong Uniqueness and High-Friction Limit for Euler-Riesz Systems

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Abstract. In this work, we employ the relative energy method to obtain a weak-strong uniqueness principle for a Euler-Riesz system, as well as to establish its convergence in the high-friction limit towards a gradient flow equation. The main technical challenge in our analysis is addressed using a specific case of a Hardy-Littlewood-Sobolev inequality for Riesz potentials.

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Key words: Euler-Riesz equations, weak-strong uniqueness, high-friction limit, relative energy method.

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1 Introduction

In this work we consider the following Euler-Riesz system in $]0,T[\times\Omega$:

$$\begin{cases} \partial_t\rho+\nabla\cdot(\rho u)=0, & (1.1\text{a})\\ \partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla\rho^\gamma+\kappa\rho\nabla W*\rho=-\nu\rho u, & (1.1\text{b})\end{cases}$$

where $0<T<\infty$ and Ω is either the d-dimensional torus $\mathbb{T}^d$ or a smooth bounded

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domain of $\mathbb{R}^d, d \geq 1$. In the case of a bounded domain, we consider the following no-flux boundary condition:

$$u \cdot n = 0 \quad \text{on} \ \ [0,T[\times \partial\Omega, \tag{1.2}$$

where n is any outward normal vector to the boundary of Ω. The density and linear velocity are denoted by ρ and u, respectively, $\rho^\gamma = p(\rho)$ is the pressure, and the interaction kernel W is given by

$$W(x) = \pm \frac{|x|^\alpha}{\alpha}, \quad -d < \alpha < 0.$$

The parameters $\gamma > 1, \kappa > 0, \nu \geq 0$ are the adiabatic exponent, the interaction strength, and the collision frequency, respectively. This system models a single-species fluid subject to attractive/repulsive interaction forces depending on the sign $\pm$.

The goal of this work is twofold. First, one obtains a weak-strong uniqueness property for system (1.1), and then one establishes the high-friction limit ($\nu \to \infty$) of system (1.1) towards the following diffusion-aggregation equation:

$$\partial_t \rho = \nabla \cdot (\nabla \rho^\gamma + \kappa \rho \nabla W * \rho) \quad \text{in} \ \ ]0,T[\times \Omega \tag{1.3}$$

with the boundary conditions (if Ω is a bounded domain),

$$(\nabla \rho^\gamma + \kappa \rho \nabla W * \rho) \cdot n = 0 \quad \text{on} \ \ [0,T[\times \partial\Omega. \tag{1.4}$$

These equations find applications in mathematical biology or plasma physics, for instance in the modelling of the behaviour of cell populations as adhesion or chemotaxis, or in the modelling of charged particles subject to electric forces, see [4–6] and references therein.

Both results are obtained using the relative energy method. In both cases, the necessary estimates are essentially identical. This serves as a clear illustration of the close mathematical relationship between weak-strong uniqueness principles and relaxation phenomena.

The relative energy method is an efficient methodology for achieving stability results, including weak-strong uniqueness principles, and establishing asymptotic limits. Since its origin, e.g. [14], this method has seen an extensive applicability to diffusive relaxation [2,3,9,17,18].

In the present manuscript, we consider weak and strong solutions for (1.1) and strong solutions for (1.3). Eq. (1.3) can be regarded as a gradient flow in the space of probability measures endowed with a Wasserstein distance [7,8]. In fact, Eq. (1.3) can be written as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \rho u = -\nabla \rho^\gamma - \kappa \rho \nabla W * \rho. \end{cases}$$