Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2024-0014

Vol. **3**, No. 3, pp. 349-368
August 2024

An Informal Account of Recent Results on Initial-Boundary Value Problems for Systems of Conservation Laws

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Received 3 May 2024; Accepted 17 June 2024

Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

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Abstract. This note aims at providing a rather informal and hopefully accessible overview of the fairly long and technical work [F. Ancona *et al.* ArXiv: 2401.14865], where new global-in-time existence results for admissible solutions of nonlinear systems of conservation laws defined in domains with boundaries are established. The main novelty of that work is that the solution is constructed by taking into account the underlying viscous mechanism, which is relevant because, in the case of initial-boundary value problems, different viscous approximations yield in general different limits. In the present note we will frame the analysis of the paper mentioned in the relevant context, compare the main result with the previous existing literature, and touch upon the most innovative technical points of the proof.

AMS subject classifications: 35L65, 35B30

Key words: Systems of conservation laws, initial-boundary value problems, hyperbolic systems, wave front-tracking, boundary characteristic case, mixed hyperbolic-parabolic systems, boundary layers.

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1 Introduction

The goal of this note is to provide an handwaving overview of some recent progress on the analysis of the initial-boundary value problem for nonlinear systems of conservation laws in the form

$$\mathbf{g}(\mathbf{v})_t + \mathbf{f}(\mathbf{v})_x = \mathbf{0}. \tag{1.1}$$

In the above expression, the unknown $\mathbf{v}$ depends on the variables (t,x), with x one-dimensional, and attains values in $\mathbb{R}^N$. The functions $\mathbf{g},\mathbf{f}: \mathbb{R}^N \to \mathbb{R}^N$ are smooth and satisfy suitable assumptions that we touch upon in the following. The exposition in this note is mainly based on the recent paper [4], where (1.1) is coupled with the viscous approximation

$$\mathbf{g}(\mathbf{v}^\varepsilon)_t + \mathbf{f}(\mathbf{v}^\varepsilon)_x = \varepsilon\big(\mathbf{D}(\mathbf{v}^\varepsilon)\mathbf{v}^\varepsilon_x\big)_x. \tag{1.2}$$

In the above expression, $\mathbf{D}$ is a positive semi-definite $N \times N$ matrix depending on the physical model under consideration. We discuss in the following the precise assumptions we impose on $\mathbf{D}$, for the time being we only mention that we rely on the analysis in the fundamental works by Kawashima and Shizuta [30]. In particular, our assumptions apply in the case of the compressible Navier-Stokes (or Navier-Stokes-Fourier) equations and the viscous magneto-hydro-dynamic (MHD) equations, which are the most natural choices for approximating the Euler and inviscid MHD equations, respectively. From the analytical standpoint, a very relevant feature of these cases (and of most, if not all, the physically relevant cases) is that the matrix $\mathbf{D}$ is singular, so that (1.2) is a mixed hyperbolic-parabolic system. This accounts for severe technical challenges that we touch upon in the following.

Coupling (1.1) with the underlying viscous mechanism is especially important in the case of initial-boundary value problems because, as we will discuss in the following, different choices of $\mathbf{D}$ in (1.2) yield in general different solutions of (1.1) in the vanishing ε limit. The main result of [4], which is also Theorem 3.1 below, establishes global-in-time existence of admissible solutions of (1.1) consistent with the underlying viscous mechanism (1.2).

The outline of this note is as follows. In Section 2, we discuss the main motivation for the analysis in [4] by providing some background information and focusing in particular on the viscous approximation of initial-boundary value problems. In Section 3, we state the main result of [4], extensively comment on it, and provide a very high level overview of the proof. In Section 4, we compare the main result of [4] with the existing literature, and finally in Section 5, we explain what we feel are, from the technical standpoint, the most interesting points of the proof of Theorem 3.1.