

A Constructive Proof of Korn's Scaled Inequalities for Shells

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Received 14 December 2023; Accepted 11 January 2024

Abstract. One of Korn's scaled inequalities for shells asserts that the H^1 -norm of a displacement field of a shell with thickness 2ε clamped on a portion of its lateral boundary, once scaled to a domain independent of ε , is bounded above by the L^2 -norm of the corresponding scaled infinitesimal strain tensor field multiplied by a constant of order ε^{-1} . We give a constructive proof to this inequality, and to other two inequalities of this type, which is thus different from the original proof of Ciarlet et al. [Arch. Rational Mech. Anal. 136 (1996), 163–190].

AMS subject classifications: 74K25, 74G45, 35B40

Key words: Korn inequalities, shells, asymptotic analysis.

1 Introduction

The notation used, but not defined, in this introduction is defined in Section 2.

Given a domain ω in \mathbb{R}^2 and a non-empty relatively open subset γ_0 of the boundary of ω , one of Korn's scaled inequality for shells asserts that there exists two constants $\varepsilon_0 = \varepsilon_0(\omega, \boldsymbol{\theta}) > 0$ and $C_0 = C_0(\omega, \boldsymbol{\theta}, \gamma_0)$ such that for all $0 < \varepsilon \leq \varepsilon_0$

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and for all vector fields $\mathbf{u} = (u_i) \in H^1(\Omega^1; \mathbb{R}^3)$, $\Omega^1 := \omega \times (-1, 1)$, that vanish on $\gamma_0 \times (-1, 1)$,

$$\|\mathbf{u}\|_{H^1(\Omega^1)} \leq \frac{C_0}{\varepsilon} \|\mathbf{e}(\varepsilon, \mathbf{u})\|_{L^2(\Omega^1)}, \quad (1.1)$$

where $\mathbf{e}(\varepsilon, \mathbf{u}) = (e_{ij}(\varepsilon, \mathbf{u})) \in L^2(\Omega^1; \mathbb{S}^3)$ is the matrix field defined at each point $(\mathbf{y}, x_3) \in \Omega^1$, $\mathbf{y} = (y_\alpha) \in \omega$ by

$$\begin{aligned} e_{\alpha\beta}(\varepsilon, \mathbf{u}) &:= \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial y_\beta} + \frac{\partial u_\beta}{\partial y_\alpha} \right) - \Gamma_{\alpha\beta}^k(\varepsilon) u_k, \\ e_{\alpha 3}(\varepsilon, \mathbf{u}) = e_{3\alpha}(\varepsilon, \mathbf{u}) &:= \frac{1}{2} \left(\frac{\partial u_3}{\partial y_\alpha} + \frac{1}{\varepsilon} \frac{\partial u_\alpha}{\partial x_3} \right) - \Gamma_{\alpha 3}^\beta(\varepsilon) u_\beta, \\ e_{33}(\varepsilon, \mathbf{u}) &:= \frac{1}{\varepsilon} \frac{\partial u_3}{\partial x_3}, \end{aligned} \quad (1.2)$$

$\Gamma_{\alpha j}^k(\varepsilon) \in C^0(\Omega^1)$ are the unique functions such that

$$\frac{\partial}{\partial y_\alpha} (\mathbf{g}_j(\varepsilon)) = \Gamma_{\alpha j}^k(\varepsilon) \mathbf{g}_k(\varepsilon),$$

where, for all $(\mathbf{y}, x_3) \in \Omega^1$,

$$\begin{aligned} \mathbf{g}_\alpha(\varepsilon)(\mathbf{y}, x_3) &:= \mathbf{a}_\alpha(\mathbf{y}) + \varepsilon x_3 \partial_\alpha \mathbf{a}_3(\mathbf{y}), \\ \mathbf{g}_3(\varepsilon)(\mathbf{y}, x_3) &:= \mathbf{a}_3(\mathbf{y}) \end{aligned}$$

with

$$\mathbf{a}_\alpha := \frac{\partial \boldsymbol{\theta}}{\partial y_\alpha} \in C^2(\bar{\omega}; \mathbb{R}^3), \quad \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \in C^2(\bar{\omega}; \mathbb{R}^3).$$

This result is essential in shell theory to obtain two-dimensional shell models from the three-dimensional model of elasticity by means of convergence theorems when the thickness 2ε of the shell go to zero, see, e.g. Ciarlet [8].

Inequality (1.1) has been proved by Ciarlet *et al.* [10, Theorem 4.1] (see also Ciarlet [8, Theorem 5.3.1]) by a contradiction argument, which we briefly sketch below. Assume that no constants ε_0 and C_0 exist such that inequality (1.1) holds for all $0 < \varepsilon \leq \varepsilon_0$ and for all vector fields $\mathbf{u} = (u_i) \in H^1(\Omega^1; \mathbb{R}^3)$ that vanish on $\gamma_0 \times (-1, 1)$. Then there exist sequences $\varepsilon_m > 0$ and $\mathbf{u}^m = (u_i^m) \in H^1(\Omega^1; \mathbb{R}^3)$, $m \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon_m &\rightarrow 0 \quad \text{when } m \rightarrow +\infty, \\ \mathbf{u}^m &= \mathbf{0} \quad \text{on } \gamma_0 \times (-1, 1), \quad \forall m \in \mathbb{N}, \\ \|\mathbf{u}^m\|_{H^1(\Omega)} &= 1, \quad \forall m \in \mathbb{N}, \\ \frac{1}{\varepsilon_m} \mathbf{e}(\varepsilon_m, \mathbf{u}^m) &\rightarrow 0 \quad \text{in } L^2(\Omega^1; \mathbb{S}^3) \quad \text{when } m \rightarrow +\infty. \end{aligned} \quad (1.3)$$