

# Some Complexes and Modules Induced by Strongly FP-injective Modules

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Received 20 September 2025; Accepted 17 November 2025

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**Abstract.** Let  $R$  be an associative ring with identity. In this paper, we consider generalizations of Gorenstein FP-injective  $R$ -modules and FP-injective complexes, give the definitions and characterizations of strongly Gorenstein FP-injective  $R$ -modules and strongly FP-injective complexes, which are induced by strongly FP-injective modules. Then we investigate the strongly FP-injective dimension of complexes.

**AMS subject classifications:** 16E05, 18G35, 18G20

**Key words:** Cotorsion pairs, strongly Gorenstein FP-injective modules, strongly FP-injective complexes, strongly FP-injective dimension.

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## 1 Introduction

Gorenstein homological theories play an important role in relative algebra. Auslander [12] introduced the notion of  $G$ -dimension of finite  $R$ -modules over a commutative Noetherian local ring. Auslander and Bridge [1] extended this notion to two sided Noetherian rings. Enochs and Jenda [3] defined Gorenstein projective modules (not necessarily finitely generated) and Gorenstein injective modules over arbitrary ring. Later, many authors have studied and generalized these notions successively.

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In 2013, as an extension of the notion of Gorenstein injective modules, Hu and Zhang [8] introduced the notion of Gorenstein FP-injective  $R$ -modules over a left coherent ring based on the complete hereditary cotorsion pair  $(\mathcal{FP}, \mathcal{FI})$  in  $R\text{-Mod}$ . Here, the symbol  $\mathcal{FP}$  (resp.,  $\mathcal{FI}$ ) stands for the subcategory of all FP-projective (resp., FI-injective) modules. Li *et al.* [9] give the definition of strongly FP-injective modules, and proved that  $({}^{\perp_1}\mathcal{SFI}, \mathcal{SFI})$  is a complete hereditary cotorsion pair over arbitrary ring, where  $\mathcal{SFI}$  denotes the subcategory of all strongly FP-injective modules. It is natural to consider another extension of Gorenstein injective modules based on the cotorsion pair  $({}^{\perp_1}\mathcal{SFI}, \mathcal{SFI})$ , that is called strongly Gorenstein FP-injective modules in this paper.

Rozas [11] systematically introduced projective complexes, injective complexes and flat complexes. Yang and Liu [16] give the definition of FP-injective complexes. Wang and Liu [17] further investigated FP-injective dimension of complexes. These works are based on a left coherent ring. One purpose of this paper is to extend above works: we introduce and study strongly FP-injective complexes and strongly FP-injective dimension of complexes over an arbitrary ring.

We now state the main results of this paper.

**Theorem 1.1.** *Let  $M$  be an  $R$ -module. Then the following statements are equivalent:*

- (1)  $M$  is a strongly Gorenstein FP-injective module.
- (2)  $M \in ({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI})^{\perp}$  and there is an exact sequence  $\cdots \rightarrow N_1 \rightarrow N_0 \rightarrow M \rightarrow 0$  with each  $N_i \in {}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}$ , which is exact under  $\text{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$ .
- (3) There is an exact sequence  $S = \cdots \rightarrow S_1 \rightarrow S_0 \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots$  with each  $S_i \in \mathcal{SFI}$  such that  $M \cong \text{Im}(S_0 \rightarrow S_{-1})$ , which is exact under  $\text{Hom}_R(\mathcal{SFI}, -)$ .
- (4) There is an exact sequence  $S = \cdots \rightarrow S_1 \rightarrow S_0 \rightarrow S_{-1} \rightarrow S_{-2} \rightarrow \cdots$  with each  $S_i \in \mathcal{SFI}$  such that  $M \cong \text{Im}(S_0 \rightarrow S_{-1})$ , which is exact under  $\text{Hom}_R({}^{\perp_1}\mathcal{SFI} \cap \mathcal{SFI}, -)$ .

**Theorem 1.2.** *Let  $X$  be a complex. Then the following conditions are equivalent:*

- (1)  $X$  is a flat complex.
- (2)  $X^+ = \underline{\text{Hom}}(X, D^1(Q/Z))$  is a strongly FP-injective complex.

**Theorem 1.3.** *Assume that  $X$  is a strongly FP-injective complex and  $F$  is a finitely presented complex. Then  $\text{Ext}^{i \geq 1}(F, X) = 0$ .*

**Theorem 1.4.** *Let  $C$  be a complex and  $n$  an integer. Then the following assertions about strongly FP-injective dimension of  $C$  are equivalent:*

- (1)  $\widetilde{\text{SFI}}\text{-id}(C) \leq n$ .