Constraint-Preserving High-Order Compact OEDG Method for Spherically Symmetric Einstein-Euler System

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Abstract. Numerical simulation of the spherically symmetric Einstein-Euler (EE) system faces severe challenges due to the stringent physical admissibility constraints of relativistic fluids and the geometric singularities inherent in metric evolution. This paper proposes a high-order constraint-preserving (CP) compact oscillation-eliminating discontinuous Galerkin (cOEDG) method specifically tailored to address these difficulties. The method integrates a scale-invariant oscillation-eliminating mechanism [M. Peng, Z. Sun, K. Wu, *Math. Comp.*, 94 (2025)] into a compact Runge-Kutta DG framework. By characterizing the convex invariant region of the hydrodynamic subsystem with general barotropic equations of state, we prove that the proposed scheme preserves physical realizability (specifically, positive density and subluminal velocity) directly in terms of conservative variables, thereby eliminating the need for complex primitive-variable checks. To ensure the geometric validity of the spacetime, we introduce a bijective transformation of the metric potentials. Rather than evolving the constrained metric components directly, the scheme advances unconstrained auxiliary variables whose inverse mapping automatically enforces strict positivity and asymptotic bounds without any limiters. The resulting CP compact OEDG (CPcOEDG) method exhibits robustness and the expected accuracy in capturing strong gravity-fluid interactions, as demonstrated by simulations of black hole accretion and relativistic shock waves.

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1　Introduction

Einstein's theory of general relativity [10] asserts that matter and energy curve spacetime. In regimes dominated by strong relativistic effects, such as black holes, neutron stars, and high-energy transients, Newtonian approximations are no longer adequate, and fluid dynamics must be reformulated on a curved space-time governed by the Einstein equations. The canonical model for this interaction is the EE system, where the relativistic Euler equations for the fluid are tightly coupled with the dynamical evolution of the spacetime metric.

The EE system is highly nonlinear, and exact solutions are rarely available in physically interesting settings. A central numerical difficulty is that smooth initial data typically evolve into discontinuous solutions (e.g., shocks), causing high-order schemes to develop spurious Gibbs-type oscillations. For the EE system, these oscillations are often spurious, and they tend to violate essential physical constraints (e.g., positivity of density and pressure, subluminal velocity), which can lead to nonlinear instabilities and eventual blow-up of the simulation. Hence, robust, high-resolution, CP numerical schemes are indispensable for accurately predicting relativistic dynamics and interpreting multimessenger observations.

The discontinuous Galerkin (DG) method, introduced by Reed and Hill [24] and systematically developed by Cockburn and Shu [3–7], provides a powerful framework for such simulations thanks to its high-order accuracy, inherent parallelism, and geometric flexibility. However, applying DG methods to the EE system raises two major challenges:

Constraint preservation. Physical constraints on both the spacetime metric and the hydrodynamic state are crucial for physical fidelity and numerical stability. These include positivity of density and pressure, subluminal velocity, and admissibility of the metric components. Violating these constraints may generate complex eigenvalues of the Jacobian, rendering the discrete problem ill-posed and causing simulation failure. For hyperbolic conservation laws, two main approaches are widely used to enforce such bounds for high-order schemes: flux-correction limiters [14,30,34], which blend high-order fluxes with first-order bound-preserving ones, and local scaling limiters [27,32,37,38], which guarantee pointwise constraints provided that the cell averages are admissible.