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The Exponential Decaying Noncommutative Differential Forms and Convolution Algebras

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Abstract. We construct an algebra of rapidly decaying C_0 functions on an étale (Lie) groupoid, which extends the standard algebra of compactly supported noncommutative differential forms. In particular, using the theory of bisections, we prove that this algebra is closed under convolution. This construction clarifies the superconnection proof of Gorokhovsky and Lott.

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1 Introduction

Gorokhovsky and Lott [4] gave a superconnection heat kernel proof, in the style of Bismut, of Connes' index theorem. They considered a smooth étale groupoid G acting on a G-proper manifold and a G-Dirac type operator D. Given a closed graded trace η on $\Omega^{\bullet}_{\omega}(G)$, they proved that

$$\langle \mathrm{ch}(\mathrm{Ind}(D)),\eta\rangle = \int_M \hat{A}(T\mathscr{F})\,\mathrm{ch}(V)\,v^*\Phi_\eta \in \mathbb{C}. \tag{1.1}$$

As pointed out in [8], Connes' index theorem for G-proper manifolds unified most of the existing index theorems at that time under a single statement.

Before delving further into the work of Gorokhovsky and Lott, we first recall some essential background on index theory. As a starting point, we recall the family index theorem, which considers a family of elliptic pseudodifferential operators that depend continuously on a parameter from some compact space B.

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The index of such a family of operators is an element in the K-theory of B. A special case occurs when the kernel and the cokernel are vector bundles. In such a case, the index is the difference of the classes of these bundles [1].

One can reformulate this situation by considering the C*-algebra $C(B)$ instead of the topology of B. For instance, a vector bundle over B is just a finitely generated projective $C(B)$-module. Any family of elliptic operators is invertible modulo fiberwise smoothing operators. Hence, its index lies in the C*-algebraic K-theory of the algebra of fiberwise smoothing operators, which is isomorphic to $K_0(B)$. For general C*-algebras, the Mishchenko-Fomenko index theorem [11] for elliptic operators on Hilbert C*-modules formalized and generalized this point of view. The index of such an elliptic operator is an element in the K-theory of the C*-algebra.

The index problem is simplified by considering the Chern character of $\mathrm{Ind}(D)$, which lies in de Rham cohomology $H^*(B)$. Bismut's elementary proof of the local family index theorem (at the level of Chern characters) used the superconnection formalism. One considers a family of generalized Laplacians with differential form coefficients, rather than the family of Dirac Laplacians $(D_z)^2_{z\in B}$. When combined, the fiber supertraces of the heat kernels corresponding to this new family of generalized Laplacians produce a closed differential form on B, known as the superconnection Chern form. It can be proven that $\mathrm{ch}(\mathrm{Ind}(D))$ equals the cohomology class of the superconnection Chern form (cf. [2, Theorem 9.33]). For more details, see [2,7,15].

Replacing $C(B)$ by the (noncommutative) convolution algebra of an étale groupoid in the superconnection formalism described above, Gorokhovsky and Lott extended the local family index theorem. According to Connes' index theorem, the index of D belongs to the K-theory group $K_0(C_r^*(G))$. Since $C_r^*(G)$ lacks a naturally dense smooth subalgebra stable under the holomorphic functional calculus, Gorokhovsky and Lott overcame this problem by defining $\mathrm{Ind}(D)$ as the K-theory group element represented by the difference between the index projection and a standard projection. In their setting, $\mathrm{Ind}(D)$ (refined index class) is an element in the K-theory of a certain algebra of smoothing $C^*(G)$-operators. They then developed homological computations and defined a graded differential algebra (GDA) that can be considered a space of noncommutative forms in the general étale groupoid case. Its cohomology generalizes the de Rham cohomology. The key ingredient of the superconnection proof is the heat kernel, which is defined by [4, (6.42)]

$$\begin{aligned} e^{-D(u)^2} &= \exp\big(-D(u)^2\big) \\ &= \exp(-\Delta_u) + \int_0^1 \exp(-\sigma_1\Delta_u)\star\mathfrak{P}\star\exp\big(-(1-\sigma_1)\Delta_u\big)d\sigma_1 \end{aligned}$$

$$+\int_0^1\int_0^{1-\sigma_1}\exp(-\sigma_1\Delta_u)\star\mathfrak{P}\star\exp(-\sigma_2\Delta_u)\star\mathfrak{P}$$
$$\star\exp\big(-(1-\sigma_1-\sigma_2)\Delta_u\big)d\sigma_2 d\sigma_1+\cdots,$$

where $\exp(-\Delta_u)$ is the fiberwise heat kernel, and $\mathfrak{P}=\Delta_u-D(u)^2$ is the lower order and noncommutative parts of the curvature of the Bismut superconnection. The fiberwise heat kernel does not have compact support but satisfies the decay condition [4, (6.34)]

$$\sup_{p,q\in Z_x} e^{Qd(p,q)}\left|e^{(-\Delta_u)}(p,q)\right|<\infty.$$

The superconnection Chern form is obtained after taking the supertrace of the heat operator.

Because of the noncompact support of the heat kernel, the authors of [4] needed to consider a rapidly decreasing algebra $\Omega^\bullet_\omega(G)$, which is a completion of $\Omega^\bullet_c(G)$. See also [9] for a special case, which is an original idea and provides more information. However, they did not prove the smoothness of their convolution product on the rapidly decreasing algebra. Even for the compactly supported algebra, they only stated the result without proof. In [3, Proposition 3.11], the proof is given that the convolution product on the compactly supported algebra is well-defined. Our motivation, therefore, is to prove the smoothness of the convolution product on the rapidly decreasing algebra, which is a GDA.

We modify the definition of the rapidly decreasing algebra $\Omega^\bullet_\omega(G)$ slightly by imposing a C_0 condition. This is natural, as it ensures that the infinite sum converges and that the resulting functions are smooth.

In the appendix of [5], Gorokhovsky and Lott showed one can replace $\Omega^\bullet_\omega(G)$ by $\Omega^\bullet_c(G)$, by using finite propagation speed methods. This allows them to establish the index theorem without employing the heat kernel or the Chern character. However, for the study of characteristic classes such as Chern-Simons class and analytic torsion form (see [10, 13, 14]), it would be necessary to consider the rapidly decreasing algebra.

2 The compactly supported algebra and bisection

We first recall the definition of a groupoid. A groupoid is a set of objects $G^{(0)}$ and a set of arrows $G^{(1)}$ endowed with the following structural maps:

- source and target maps $s,t:G^{(1)}\to G^{(0)}$,
- multiplication map $m:G^{(2)}=\{(\gamma_1,\gamma_2)\in G\times G: s(\gamma_1)=t(\gamma_2)\}\to G^{(1)}$,

- unit $u: G^{(0)} \to G^{(1)}$,
- inverse $i: G^{(1)} \to G^{(1)}$.

These maps are also often denoted more suggestively by $u(x)=1_x, i(\gamma)=\gamma^{-1}$ and $m(\gamma_1,\gamma_2)=\gamma_1\gamma_2$. Additionally, these structural maps are required to satisfy the following group-like axioms:

- $s(1_x)=t(1_x)=x$,
- $1_{t(\gamma)}\gamma=\gamma 1_{s(\gamma)}=\gamma$,
- $s(\gamma^{-1})=t(\gamma)$ and $t(\gamma^{-1})=s(\gamma)$,
- $\gamma\gamma^{-1}=1_{t(\gamma)}$ and $\gamma^{-1}\gamma=1_{s(\gamma)}$,
- $s(\gamma_1\gamma_2)=s(\gamma_2)$ and $t(\gamma_1\gamma_2)=t(\gamma_1)$, whenever $(\gamma_1,\gamma_2)\in G^{(2)}$,
- associativity: $\gamma_1(\gamma_2\gamma)=(\gamma_1\gamma_2)\gamma$, whenever $(\gamma_1,\gamma_2),(\gamma_2,\gamma)\in G^{(2)}$,

for any $\gamma_1,\gamma_2,\gamma\in G$ and $x\in G^{(0)}$.

A topological groupoid is a groupoid with a locally compact topology with respect to which both the multiplication and the inversion are continuous. A Lie groupoid (or smooth groupoid) is a topological groupoid for which all the structural maps are smooth, and the source and target maps $s,t: G\to G^{(0)}$ are smooth submersions.

A groupoid is étale if s and t are local diffeomorphisms. In this work, we assume all groupoids are Lie, Hausdorff, and étale groupoids.

We briefly recall the definition of Gorokovsky and Lott's convolution algebra and GDA structure (cf. [4]). In addition, we obtain a star GDA by defining a suitable involution.

Definition 2.1. *Let $G \rightrightarrows G^{(0)}$ be an étale groupoid. Let $C_c^\infty(G)$ denote the natural convolution algebra of smooth functions of compact support on G. Let $C_c^\infty(G)$ be equipped with multiplication and involution*

$$
\begin{aligned}
(f_1 \star f_2)(\gamma_0) &:= \sum_{\gamma\gamma'=\gamma_0} f_1(\gamma) f_2(\gamma'),\\
f^*(\gamma) &:= \overline{f(\gamma^{-1})}.
\end{aligned}
$$

Let

$$G^{(n)}:=\Big\{(\gamma_1,\cdots,\gamma_n)\in G^n:\ t(\gamma_{i+1})=s(\gamma_i),\ i=1,\ldots,n-1\Big\}.$$

It is a manifold of the same dimension as G. Let $\Omega_c^{m,n}(G)$ be the quotient of $\Omega_c^m(G^{(n+1)})$ by the forms that are supported on $\{(\gamma_0,\cdots,\gamma_n):\gamma_j$ is a unit for some $j>0\}$. We define $t_n:G^{(n)}\to G^{(0)}$ and $s_n:G^{(n)}\to G^{(0)}$ by $t_n(\gamma_1,\cdots,\gamma_n)=t(\gamma_1)$ and $s_n(\gamma_1,\cdots,\gamma_n)=s(\gamma_n)$, respectively. They are also local homeomorphisms and induce isomorphisms between tangent spaces.

The following definition is parallel to that of [13, Definition 3.4], (see also [4, (6.3)] and [10, Section 4]).

Definition 2.2. *The vector space $\Omega_c^{m,n}(G)$ of noncommutative de Rham differential forms is equipped with multiplication and involution as follows. If $\omega_1\in\Omega_c^{m_1,k}(G)$ and $\omega_2\in\Omega_c^{m_2,l}(G)$, then $\omega_1\star\omega_2\in\Omega_c^{m_1+m_2,k+l}(G)$ is defined by*

$$\begin{aligned}
&(\omega_1\star\omega_2)(\gamma_0;\cdots,\gamma_{k+l})\\
:=&\sum_{\gamma\gamma'=\gamma_k}\omega_1(\gamma_0;\cdots,\gamma)\wedge(t\circ s^{-1})\omega_2(\gamma';\cdots,\gamma_{k+l})\\
&-\sum_{\gamma\gamma'=\gamma_{k-1}}\omega_1(\gamma_0;\cdots,\gamma,\gamma')\wedge(t\circ s^{-1})\omega_2(\gamma_k;\cdots,\gamma_{k+l})\\
&+\cdots+(-1)^k\sum_{\gamma\gamma'=\gamma_0}\omega_1(\gamma;\gamma',\cdots,\gamma_{k-1})\wedge(t\circ s^{-1})\omega_2(\gamma_k;\cdots,\gamma_{k+l}).
\end{aligned}$$

In forming the wedge product, the map $t\circ s^{-1}$ is used to identify cotangent spaces. For $\omega_2(\gamma_k;\cdots,\gamma_{k+l})\in\Lambda^{m_2}T^*_{t(\gamma_k)}G^{(0)}$, if we fix a $(\gamma,\gamma',\cdots,\gamma_{k-1})\in G^{(k+1)}$, there is an open neighbourhood U of $\gamma_0\cdots\gamma_{k-1}$ such that the restrictions of t and s to U are diffeomorphisms. Thus, $t\circ s^{-1}$ identifies $\Lambda^{m_2}T^*_{t(\gamma_k)}G^{(0)}$ and $\Lambda^{m_2}T^*_{t(\gamma_0)}G^{(0)}$.

In addition, we define the involution

$$\begin{aligned}
\omega^*(\gamma_0;\cdots,\gamma_k):=&\sum_{\gamma\gamma'=\gamma_0^{-1}}(t\circ s^{-1})\overline{\omega}(\gamma_k^{-1};\cdots,\gamma_1^{-1},\gamma)\chi_{G^{(0)}}(\gamma')\\
&-\sum_{\gamma\gamma'=\gamma_1^{-1}}(t\circ s^{-1})\overline{\omega}(\gamma_k^{-1};\cdots,\gamma,\gamma')\chi_{G^{(0)}}(\gamma_0^{-1})\\
&+\cdots+(-1)^k\sum_{\gamma\gamma'=\gamma_k^{-1}}(t\circ s^{-1})\overline{\omega}(\gamma;\gamma',\cdots,\gamma_1^{-1})\chi_{G^{(0)}}(\gamma_0^{-1}).
\end{aligned}$$

Lemma 2.1. *The involution defined above satisfies*

$$\begin{aligned}
&\omega^{**}(\gamma_0;\cdots,\gamma_k)=\omega(\gamma_0;\cdots,\gamma_k),\\
&(\omega_1\star\omega_2)^*(\gamma_0;\cdots,\gamma_{k+l})=(\omega_2^*\star\omega_1^*)(\gamma_0;\cdots,\gamma_{k+l}).
\end{aligned}$$

Proof. The proof follows by straightforward calculations. □

Let d_1 be the de Rham differential on $\Omega_c^{*,*}(G)$ and define $d_2:\Omega_c^{*,*}(G)\to\Omega_c^{*,*+1}(G)$ by

$$(d_2\omega)(\gamma_0;\gamma_1,\cdots,\gamma_n):=\chi_{G^{(0)}}(\gamma_0)\omega(\gamma_1;\cdots,\gamma_n).$$

From the total complex of $\Omega_c^{*,*}(G)$, we obtain a star GDA and denote it by $\Omega_c^{\bullet}(G):=\bigoplus_{m,n\geq 0}\Omega_c^{m,n}(G)$.

In the following counterexample, we illustrate the importance of considering compactly supported functions in the above definitions.

Example 2.1. Let $\bigcup_i U_i$ be a cover of M. Then we define the *Čech* groupoid G by

$$G^{(0)}=\bigsqcup_i U_i,\quad G=\bigsqcup_{ij} U_{ij},$$

where $U_{ij}=U_i\cap U_j$. Define the target map $t:U_{ij}\to U_i$, where for any $\gamma\in U_{ij}$, $t(\gamma)=\gamma$ but the right-hand side is regarded as an element of U_i. The source map $s:U_{ij}\to U_j$ is defined similarly. For $\gamma_1\in U_{ij},\gamma_2\in U_{jk}$ satisfying $t(\gamma_2)=s(\gamma_1)$, hence $\gamma_1=\gamma_2$ as an element in U_j, the multiplication $\gamma_1\gamma_2=\gamma_1=\gamma_2$ but as an element in U_{ik}. The unit map is the identity map from U_i to $U_{ii}=U_i\cap U_i$. The inverse map is the identity on $U_{ij}=U_i\cap U_j=U_j\cap U_i=U_{ji}$.

We consider $M=\mathbb{R}$ with the open cover $U_1=\{x\in\mathbb{R}:x<1\}$ and $U_2=\{x\in\mathbb{R}:x>-1\}$. Thus, we have $G=U_{11}\sqcup U_{12}\sqcup U_{21}\sqcup U_{22}$. Then U_{ij} are all connected components of G. Let

$$f(\gamma)=\begin{cases}1, & \text{if}\quad \gamma\in U_{12},\\ 0, & \text{otherwise},\end{cases}\qquad g(\gamma)=\begin{cases}1, & \text{if}\quad \gamma\in U_{21},\\ 0, & \text{otherwise}.\end{cases}$$

Obviously, $f,g\in C^\infty(G)$. However, we have

$$(f\star g)(\gamma)=\begin{cases}1, & \text{if}\quad \gamma\in U_{12}U_{21}=(-1,1)\subseteq U_{11},\\ 0, & \text{otherwise},\end{cases}$$

which is not continuous since $(f\star g)(U_{11})=(f\star g)(-\infty,1)$ is disconnected. Hence, $f\star g\notin C^\infty(G)$.

If G is the crossed product groupoid, its connected components are $\{h\}\times M$, which are bisections. On $\{h\}\times M$, the source and target maps are surjective. Since there is no need to extend by 0, $f\star g$ is always smooth.

The counterexample above explains why we add the C_0 condition on bisections to our algebras. In the following, we briefly review the theory of bisections, primarily based on [3,12].

Definition 2.3. *An open subset $U\subseteq G$ is called a bisection if the restrictions of s and t to U are injective.*

As pointed out in [3], an étale groupoid can be defined by its bisections (called slices in that paper). Let U and V denote open subsets in G. Let $U^{-1}=\{u^{-1}:u\in U\}$ and $UV=\{uv:u\in U,v\in V,(u,v)\in G^{(2)}\}$ (possibly empty).

Proposition 2.1. *We list some basic facts about bisection:*

(i) *G has a basis of bisections (cf.* [3, *Proposition* 3.5], [12, *Lemma* 8.4.9]*).*

(ii) *If U and V are bisections, then U^{-1} and UV are bisections (cf.* [3, *Proposition* 3.8]*).*

(iii) *For $f_1,f_2\in C_c^\infty(G)$, if $\operatorname{supp}(f_1)\subseteq U$ and $\operatorname{supp}(f_2)\subseteq V$, then $\operatorname{supp}(f_1^*)\subseteq U^{-1}$ and $\operatorname{supp}(f_1\star f_2)\subseteq UV$ (cf.* [12, *Lemma* 9.1.4], [3, *Proposition* 3.11]*).*

(iv) *$C_c^\infty(G)=\operatorname{span}\{f\in C_c^\infty(G)\,|\operatorname{supp}(f)$ is in a bisection$\}$ (cf.* [3, *Proposition* 3.10], [12, *Lemma* 9.1.3]*).*

Recall that $s_2,t_2:G^{(2)}\to G^{(0)}$ are defined by $s_2(\gamma_0,\gamma_1)=s(\gamma_1)$ and $t_2(\gamma_0,\gamma_1)=t(\gamma_0)$. Set $U\hat{\times}V:=(U\times V)\cap G^{(2)}=\{(u,v):u\in U,v\in V,(u,v)\in G^{(2)}\}$ (possibly empty).

Definition 2.4. *An open subset $U\subseteq G^{(2)}$ is called a bisection in $G^{(2)}$ if the restrictions of s_2 and t_2 to U are injective.*

Lemma 2.2. *If $U,V\subseteq G$ are bisections, then $U\hat{\times}V\subseteq G^{(2)}$ is a bisection in $G^{(2)}$.*

Proof. Since the restriction of the multiplication map $m|_{U\hat{\times}V}:U\hat{\times}V\to UV$ is continuous, open, and injective, it is a diffeomorphism. So $U\hat{\times}V$ is an open subset of $G^{(2)}$. If $U\hat{\times}V$ is not empty, there exists $(\gamma_u,\gamma_v)\in U\hat{\times}V$ such that $t(\gamma_v)=s(\gamma_u)$. Since V is a bisection, thus $\gamma_v=(t|_V)^{-1}(s(\gamma_u))$ and therefore $(\gamma_u,\gamma_v)=(\gamma_u,(t|_V)^{-1}(s(\gamma_u)))$, where we are denoting by $t|_V$ the restriction of t to V.

For any $(\gamma_0,(t|_V)^{-1}(s(\gamma_0)))\in U\hat{\times}V$ and $(\gamma_1,(t|_V)^{-1}(s(\gamma_1)))\in U\hat{\times}V$, suppose

$$t_2\big(\gamma_0,(t|_V)^{-1}(s(\gamma_0))\big)=t_2\big(\gamma_1,(t|_V)^{-1}(s(\gamma_1))\big),$$

which implies $t(\gamma_0)=t(\gamma_1)$. Since $t|_U$ is a diffeomorphism, we get $\gamma_0=\gamma_1$, it follows that

$$\big(\gamma_0,(t|_V)^{-1}(s(\gamma_0))\big)=\big(\gamma_1,(t|_V)^{-1}(s(\gamma_1))\big)$$

and therefore $t_2|_{U\hat{\times}V}$ is injective. Similarly, $s_2|_{U\hat{\times}V}$ is injective. □

Lemma 2.3. *The manifold* $G^{(2)}$ *has a base of bisections.*

Proof. If $(\gamma_0,\gamma_1)\in G^{(2)}$, by Proposition 2.1(i), there exist $U,V\subseteq G$ that are bisections such that $\gamma_0\in U,\gamma_1\in V$. Then (γ_0,γ_1) is contained in the bisection $U\hat{\times}V$. □

Next, we extend Proposition 2.1(iv) to the case of noncommutative forms.

Proposition 2.2. *If* $\omega\in\Omega_c^{*,1}(G)=\Omega_c^*(G^{(2)})/\sim$, *then* $\omega=\sum_{i=1}^n\omega_i$, *where* $\mathrm{supp}(\omega_i)\subseteq U_i\hat{\times}V_i$.

Proof. Fix $\omega\in\Omega_c^{*,1}(G)$. By Lemma 2.3, we can cover $\mathrm{supp}(\omega)$ with bisections $\{U_i\hat{\times}V_i:U_i,V_i\subseteq G$ are bisections$\}$, and then use compactness to pass to a finite subcover $U_1\hat{\times}V_1,\cdots,U_n\hat{\times}V_n$. Choose a partition of unity $\{\varphi_i\}$ on $\mathrm{supp}(\omega)$ subordinate to the $U_i\hat{\times}V_i$. The pointwise products $\omega_i=\omega\cdot\varphi_i$ belong to $\Omega_c^{*,1}(G)$ with $\mathrm{supp}(\omega_i)\subseteq U_i\hat{\times}V_i$, and we have $\omega=\sum_{i=1}^n\omega_i$. □

From Proposition 2.2, we have

Lemma 2.4. *If* $f\in C_c^\infty(G)$ *and* $\omega\in\Omega_c^{*,1}(G)$, *then* $f\star\omega\in\Omega_c^{*,1}(G)$ *and* $\omega\star f\in\Omega_c^{*,1}(G)$.

Proof. By using partitions of unity, we have $f=\sum_{i=1}^n f_i$ and $\omega=\sum_{h=1}^m\omega_h$. We may suppose $\mathrm{supp}(f_i)\subseteq U_i$ and $\mathrm{supp}(\omega_h)\subseteq V_j\hat{\times}W_k$, where U_i,V_j, and W_k are bisections.

Let

$$(f_i\cdot\omega_h)(a,b,c):=f_i(a)\omega_h(b;c)$$

for any $(a,b,c)\in U_i\hat{\times}V_j\hat{\times}W_k$. Thus, for $(\gamma_0,\gamma_1)\in U_iV_j\hat{\times}W_k$, we have

$$(f_i\star\omega_h)(\gamma_0;\gamma_1)=(f_i\cdot\omega_h)\big(m_{i,j}^{-1}(\gamma_0),\gamma_1\big)=(f_i\cdot\omega_h)\big(m_{i,j}^{-1},1\big)(\gamma_0,\gamma_1),$$

where $m_{i,j}$ is the restriction of the multiplication map to $U_i\hat{\times}V_j$. Then $f_i\star\omega_h=(f_i\cdot\omega_h)(m_{i,j}^{-1},1)$. It follows that $f_i\star\omega_h$ is smooth on $U_iV_j\hat{\times}W_k$. Furthermore, one easily sees that

$$(f_i\star\omega_h)(\gamma_0;\gamma_1)=\begin{cases}(f_i\cdot\omega_h)\big(m_{i,j}^{-1}(\gamma_0),\gamma_1\big), & \text{if}\quad(\gamma_0,\gamma_1)\in U_iV_j\hat{\times}W_k,\\ 0, & \text{otherwise}.\end{cases}$$

So we must only show that $f_i\star\omega_h$ is compactly supported on $U_iV_j\hat{\times}W_k$.

Let $A \subseteq U_i$ be the compact support of f_i and let $B = pr_1(\mathrm{supp}(\omega_h))$ and $C = pr_2(\mathrm{supp}(\omega_h))$. They are both compact. Then we claim that $AB \hat{\times} C$ is compact. Since $G^{(2)}$ is closed in $G \times G$, so $(AB \times C) \cap G^{(2)}$ is closed in $(AB \times C) \cap (G \times G) = AB \times C$, and hence compact.

Next, we prove that $f_i \star \omega_h$ vanishes outside $AB \hat{\times} C$. We only need to verify that $f_i \cdot \omega_h$ vanishes outside $A \hat{\times} B \hat{\times} C$. This is obvious. We deduce that $f_i \star \omega_h \in \Omega_c^*(U_i V_j \hat{\times} W_k)$. Finally, by using Proposition 2.2,

$$f \star \omega = \sum_{i=1}^{n} \sum_{h=1}^{m} f_i \star \omega_h \in \Omega_c^{*,1}(G).$$

As for $\omega \star f$, a similar argument likes $f \star \omega$ works. Let

$$(\omega_h \cdot f_i)(a,b,c) := \omega_h(a;b) f_i(c)$$

for any $(a,b,c) \in V_j \hat{\times} W_k \hat{\times} U_i$. Then we observe that

$$(\omega_h \star f_i)(\gamma_0;\gamma_1) = \begin{cases} (\omega_h \cdot f_i)\big(\gamma_0, m_{k,i}^{-1}(\gamma_1)\big), & \text{if} \quad (\gamma_0,\gamma_1) \in V_j \hat{\times} W_k U_i, \\ -(\omega_h \cdot f_i)\big(m_{j,k}^{-1}(\gamma_0),\gamma_1\big), & \text{if} \quad (\gamma_0,\gamma_1) \in V_j W_k \hat{\times} U_i, \\ 0, & \text{otherwise.} \end{cases}$$

The remaining details are omitted. □

3 The rapidly decreasing algebra

In order to define the $C_\omega^\infty(G)$ and $\Omega_\omega^\bullet(G)$, we need some assumptions on G.

Definition 3.1. *A length function on an étale groupoid G is a map $l: G \to \mathbb{R}^+$, satisfying*

(i) $l(\gamma_u) = 0$, for all $\gamma_u \in G^{(0)}$,

(ii) $l(\gamma^{-1}) = l(\gamma)$, for all $\gamma \in G$,

(iii) $l(\gamma_1\gamma_2) \leq l(\gamma_1) + l(\gamma_2)$, for all $(\gamma_1,\gamma_2) \in G^{(2)}$.

For $x \in G^{(0)}$, let $G_x = s^{-1}(x)$ and $G^x = t^{-1}(x)$. Similar to [8, (47)] and [6, Definition 3.1], we need the following

Definition 3.2. *We say that G is of exponential growth with respect to a length function l on G, if there exists $C\in\mathbb{R}^+$ such that*

$$|B(R)|:=\sup_{x\in G^{(0)}}\#\{\gamma\in G_x : l(\gamma)\leq R\}=\sup_{x\in G^{(0)}}\#\{\gamma\in G^x : l(\gamma)\leq R\}\leq e^{CR}$$

for every $R\geq 0$.

Similar growth conditions were used in the proofs of [8, Lemma 7] (in the crossed product case) and [6, Proposition 3.5].

Let $\{U_i\}_{i=1}^{\infty}\subseteq G$ be connected components and $G\subseteq\bigcup_i^{\infty}U_i$. We assume that each connected component is a bisection.

Definition 3.3. *We define all the connected components of G to be of exponential growth with respect to a length function if there exists $C\in\mathbb{R}^+$ such that*

$$\#\big\{U_i\subseteq G : \inf\{l(\gamma):\gamma\in U_i\}\leq R\big\}\leq e^{CR}. \tag{3.1}$$

Eq. (3.1) implies that G has a proper length function.

A function $f\in C^{\infty}(G)$ is said to vanish at infinity if, for every $\varepsilon>0$, there is a compact set K such that $|f(x)|<\varepsilon$, for all x not in K. We denote all such functions on G by $C_0^{\infty}(G)$. In the more general case, $\Omega_0^m(G^{(n+1)})$ can be defined similarly.

Notation 3.1. In the rest of this paper, we always use the notation

$$f_i(\gamma)=\begin{cases} f|_{U_i}(\gamma), & \text{if} \quad \gamma\in U_i,\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.1. *If $f_i,f_j\in C_0^{\infty}(G)$, $\mathrm{supp}(f_i)\subseteq U_i$, and $\mathrm{supp}(f_j)\subseteq V_j$, then $f_i\star f_j\in C_0^{\infty}(G)$ and $\mathrm{supp}(f_i\star f_j)\subseteq U_iV_j$.*

Proof. For any $\varepsilon>0$, we can find $\widehat{f_i}\in C_c^{\infty}(U_i)$ and $\widehat{f_j}\in C_c^{\infty}(V_j)$ such that

$$\|f_i-\widehat{f_i}\|_{sup}<\varepsilon/2, \quad \|f_j-\widehat{f_j}\|_{sup}<\varepsilon/2.$$

Since

$$\begin{aligned}\|f_i\star f_j\|_{sup}&=\sup_{\gamma_0\in G}|(f_i\star f_j)(\gamma_0)|\\ &=\sup_{\gamma_0\in U_iV_j}\left|f_i(t_i^{-1}t(\gamma_0))f_j(s_j^{-1}s(\gamma_0))\right|\\ &\leq\|f_i\|_{sup}\|f_j\|_{sup},\end{aligned}$$

it follows that

$$\begin{aligned}\|f_i\star f_j-\widehat{f}_i\star\widehat{f}_j\|_{sup}&=\big\|(f_i-\widehat{f}_i)\star f_j+\widehat{f}_i\star(f_j-\widehat{f}_j)\big\|\\&\leq\|f_i-\widehat{f}_i\|\|f_j\|+\|\widehat{f}_i\|\|f_j-\widehat{f}_j\|<C\varepsilon.\end{aligned}$$

Using Proposition 2.1(iii), we have $\widehat{f}_i\star\widehat{f}_j\in C_c^\infty(U_iV_j)$. Thus, $f_i\star f_j\in C_0^\infty(U_iV_j)$. □

In the rest of this paper, we define ω_h as the restriction of ω to the connected component W_h of $G^{(2)}$, such that $pr_1(W_h)\subseteq U_i$ and $pr_2(W_h)\subseteq V_j$, where U_i and V_j are both connected components of G and extend it to $G^{(2)}$ by zero.

For GDA, we have a result similar to Lemma 3.1. In fact, we only need to prove the special case.

Corollary 3.1. *If ω_h and f_i are C_0, then $\omega_h\star f_i$ is C_0.*

Proof. The proof follows an argument similar to that of Lemma 3.1, relying on Lemma 2.4. □

With the above preparation, we arrive at the central definitions of this paper.

Definition 3.4. *We define $C_\omega^\infty(G)$ as $f\in C^\infty(G)$ such that*

(i) the restrictions of f to connected components belong to $C_0^\infty(G)$,

(ii) for any $q\in\mathbb{N}$,

$$\sup_{\gamma\in G}e^{ql(\gamma)}|f(\gamma)|<\infty,$$

along with the analogous property for derivatives.

Definition 3.5. *Let $\tilde{\Omega}_\omega^{m,n}(G)$ be the subset of smooth differential forms $\omega\in\Omega^m(G^{(n+1)})$ such that*

(i) the restrictions of ω to connected components belong to $\Omega_0^m(G^{(n+1)})$,

(ii) for any $q\in\mathbb{N}$,

$$\sup_{(\gamma_0,\cdots,\gamma_n)\in G^{(n+1)}}e^{q(l(\gamma_0)+\cdots+l(\gamma_n))}|\omega(\gamma_0;\cdots,\gamma_n)|<\infty,$$

along with the analogous property for derivatives.

We denote by $\Omega_{\omega}^{m,n}(G)$ the quotient of $\tilde{\Omega}_{\omega}^{m,n}(G)$ by the forms which are supported on $\{(\gamma_0,\cdots,\gamma_n):\gamma_j \text{ is a unit for some } j>0\}$. Let $\Omega_{\omega}^{\bullet}(G):=\bigoplus_{m,n\geq 0}\Omega_{\omega}^{m,n}(G)$.

By the decay conditions, we can show that $C_{\omega}^{\infty}(G)$ is a convolution algebra and $\Omega_{\omega}^{\bullet}(G)$ is a star GDA with the same formal calculations as Definitions 2.1 and 2.2 (compare with [9, Proposition 3]). For more details about the rapidly decreasing algebra, one can see [4, p. 190] and [8, p. 221].

Theorem 3.1. *Let $f=\sum_{i=1}^{\infty}f_i$ and $g=\sum_{j=1}^{\infty}g_j$ belong to $C_{\omega}^{\infty}(G)$. Then $f\star g\in C_{\omega}^{\infty}(G)$.*

Proof. We abbreviate $t|_{U_i}$ as t_i. Observe that

$$(f\star g)(\gamma_0)=\sum_{i,j}(f\star g)_{ij}(\gamma_0)=\sum_{i,j}f_i\big(t_i^{-1}t(\gamma_0)\big)g_j\big(s_j^{-1}s(\gamma_0)\big).$$

For $q=[C]+2$ (a bound ensuring convergence of the geometric series below), we verify finiteness

$$\begin{aligned}
\sum_{i,j}\|(f\star g)_{ij}\| &=\sum_{i,j}\sup_{\gamma_0\in G}\left|f\big(t_i^{-1}t(\gamma_0)\big)g\big(s_j^{-1}s(\gamma_0)\big)\right| \\
&=\sum_{n=0}^{\infty}\sum_{n\leq \inf U_i\leq n+1}\sum_{m=0}^{\infty}\sum_{m\leq \inf V_j\leq m+1}\sup_{\gamma_0\in G}\left|f\big(t_i^{-1}t(\gamma_0)\big)g\big(s_j^{-1}s(\gamma_0)\big)\right| \\
&\leq \sum_{n=0}^{\infty}e^{C(n+1)}\sum_{m=0}^{\infty}e^{C(m+1)}e^{-q(n+m)} \\
&\leq \sum_{n=0}^{\infty}e^{([C]+1)(n+1)-qn}\sum_{m=0}^{\infty}e^{([C]+1)(m+1)-qm} \\
&\leq C_1\sum_{n=0}^{\infty}e^{-n}\sum_{m=0}^{\infty}e^{-m}<\infty.
\end{aligned}$$

Thus, $f\star g\in C_0^{\infty}(G)$ is well-defined. It follows that the restrictions of $f\star g$ to connected components belong to $C_0^{\infty}(G)$. We next show that $f\star g$ satisfies the decay condition. For any $q\in\mathbb{N}$,

$$\begin{aligned}
&\sup_{\gamma_0\in G}e^{ql(\gamma_0)}\left|(f\star g)(\gamma_0)\right| \\
=&\sup_{\gamma_0\in G}e^{ql(\gamma_0)}\left|\sum_{i,j}f\big(t_i^{-1}t(\gamma_0)\big)g\big(s_j^{-1}s(\gamma_0)\big)\right| \\
\leq&\sup_{\gamma_0\in G}\sum_{i,j}e^{q(l(t_i^{-1}t(\gamma_0))+l(s_j^{-1}s(\gamma_0)))}\left|f\big(t_i^{-1}t(\gamma_0)\big)g\big(s_j^{-1}s(\gamma_0)\big)\right|
\end{aligned}$$

$$\leq \sup_{\gamma_0\in G}\left(\sum_i e^{ql(t_i^{-1}t(\gamma_0))}\left|f\left(t_i^{-1}t(\gamma_0)\right)\right|\right)\left(\sum_j e^{ql(s_j^{-1}s(\gamma_0))}\left|g\left(s_j^{-1}s(\gamma_0)\right)\right|\right).$$

For the first factor, we have

$$\begin{aligned}
&\sup_{\gamma_0\in G}\sum_i e^{ql(t_i^{-1}t(\gamma_0))}\left|f\left(t_i^{-1}t(\gamma_0)\right)\right| \\
\leq &\sum_{n=0}^{\infty} e^{C(n+1)}e^{(q-q_1)n}, \quad \text{let } q_1=q+[C]+2 \\
\leq &C_1\sum_{n=0}^{\infty} e^{-n}<\infty.
\end{aligned}$$

The other part is similar. □

Theorem 3.1 motivates the following result, which also holds for higher-order differential forms.

Theorem 3.2. *If $\omega\in\Omega_{\omega}^{*,1}(G)$ and $f\in C_{\omega}^{\infty}(G)$, then $\omega\star f\in\Omega_{\omega}^{*,1}(G)$.*

Proof. To prove that the restrictions of $\omega\star f$ to connected components are C_0, we observe

$$\sum_{i,h}\|\omega_h\star f_i\|=\sum_{i,h}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|(\omega_h\cdot f_i)\left(m_{j,k}^{-1}(\gamma_0),\gamma_1\right)-(\omega_h\cdot f_i)\left(\gamma_0,m_{k,i}^{-1}(\gamma_1)\right)\right|.$$

Then we just need to verify for $q=[C]+2$,

$$\begin{aligned}
&\sum_{i,h}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|(\omega_h\cdot f_i)\left(m_{j,k}^{-1}(\gamma_0),\gamma_1\right)\right| \\
\leq &\sum_{i,j,k}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|\omega_h(m_{j,k}^{-1}(\gamma_0))\right|\cdot|f_i(\gamma_1)| \\
= &\sum_{n=0}^{\infty}\sum_{n\leq\inf V_j\leq n+1}\sum_{m=0}^{\infty}\sum_{m\leq\inf W_k\leq m+1}\sum_{l=0}^{\infty}\sum_{l\leq\inf U_i\leq l+1}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|\omega_h(m_{j,k}^{-1}(\gamma_0))\right|\cdot|f_i(\gamma_1)| \\
\leq &\sum_{n=0}^{\infty}e^{C(n+1)}\sum_{m=0}^{\infty}e^{C(m+1)}e^{-q(n+m)}\sum_{l=0}^{\infty}e^{C(l+1)}e^{-ql} \\
\leq &\sum_{n=0}^{\infty}e^{([C]+1)(n+1)-qn}\sum_{m=0}^{\infty}e^{([C]+1)(m+1)-qm}\sum_{l=0}^{\infty}e^{([C]+1)(l+1)-ql} \\
\leq &C_1\sum_{n=0}^{\infty}e^{-n}\sum_{m=0}^{\infty}e^{-m}\sum_{l=0}^{\infty}e^{-l}<\infty
\end{aligned}$$

and

$$\begin{aligned}&\sum_{i,h}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|(\omega_h\cdot f_i)\left(\gamma_0,m_{k,i}^{-1}(\gamma_1)\right)\right|\\
&\leq\sum_{l=0}^{\infty}\sum_{l\leq\inf V_j\leq l+1}\sum_{n=0}^{\infty}\sum_{n\leq\inf W_k\leq n+1}\sum_{m=0}^{\infty}\sum_{m\leq\inf U_i\leq m+1}\sup_{(\gamma_0,\gamma_1)\in G^{(2)}}\left|(\omega_h\cdot f_i)\left(\gamma_0,m_{k,i}^{-1}(\gamma_1)\right)\right|\\
&\leq\sum_{l=0}^{\infty}e^{C(l+1)}\sum_{n=0}^{\infty}e^{C(n+1)}e^{-q(l+n)}\sum_{m=0}^{\infty}e^{C(m+1)}e^{-qm}\\
&\leq\sum_{l=0}^{\infty}e^{([C]+1)(l+1)-ql}\sum_{n=0}^{\infty}e^{([C]+1)(n+1)-qn}\sum_{m=0}^{\infty}e^{([C]+1)(m+1)-qm}\\
&\leq C_1\sum_{l=0}^{\infty}e^{-l}\sum_{n=0}^{\infty}e^{-n}\sum_{m=0}^{\infty}e^{-m}<\infty.\end{aligned}$$

Thus, $\omega\star f=\sum_{i,h}\omega_h\star f_i\in\Omega_0^{*,1}(G)$. The proof that $\omega\star f$ satisfies the decay condition is similar to that of Theorem 3.1. □

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