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Boundedness for Commutators of Approximate Identities on Weighted Morrey Spaces

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Abstract: The aim of this paper is to set up the weighted norm inequalities for commutators generated by approximate identities from weighted Lebesgue spaces into weighted Morrey spaces.

Key words: approximate identity, weighted Morrey space, weighted BMO space, commutator

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1 Introduction

Suppose that $\varphi \in L^1(\mathbf{R}^n)$, $f \in L^p(\mathbf{R}^n)$ $(1 \le p < \infty)$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for all $\varepsilon > 0$. If the operator

$$T_\varphi f(x) = f \star \varphi_\varepsilon(x) \Longrightarrow f,$$

then as $\varepsilon \to 0$, φ_ε is called the kernel of approximate identities on $L^p(\mathbf{R}^n)$, and T_φ is called the operator of approximate identities. If φ_ε further satisfies

$$|\varphi(x-y) - \varphi(x)| \le \frac{|y|}{|x|^{n+1}}, \qquad |x| > 2|y|, \tag{1.1}$$

Francia *et al.*[1] have proved that T_φ is bounded from $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ with $1 < p < \infty$.

Recall the definitions of Muckenhoupt classes (see [2]):

$$A_p : \sup_B \left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \le C, \qquad 1 < p < \infty;$$

$$A_1 : Mw(x) \le Cw(x);$$

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$$A_\infty = \bigcup_{p>1} A_p.$$

Here B denotes any ball in $\mathbf{R}^n$, $\dfrac{1}{p}+\dfrac{1}{p'}=1$ and M is the Hardy-littlewood maximal function:

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y.$$

For a measure ν, we say $w(x)\in A_p(\nu)$ if

$$\sup_B\left(\frac{1}{\nu(B)}\int_B w\mathrm{d}\nu\right)\left(\frac{1}{\nu(B)}\int_B w^{1-p'}\mathrm{d}\nu\right)^{p-1}\le C, \qquad 1<p<\infty,$$

$$A_\infty(\nu) = \bigcup_{p>1} A_p(\nu),$$

where $\nu(B)=\displaystyle\int_B \nu$. Here and subsequently, C denotes a positive constant which may vary from line to line but will remain independent of the relevant quantities.

For $w\in A_p$, the weighted BMO space is defined by

$$\mathrm{BMO}(w) = \left\{b : \|b\|_{\mathrm{BMO}(w)} = \frac{1}{w(B)}\int_B |b(x)-b_B|\mathrm{d}x<\infty\right\},$$

where $b_B = \dfrac{1}{|B|}\displaystyle\int_B b(x)\mathrm{d}x$. Then the commutators generated by T_φ and $b\in\mathrm{BMO}(w)$ can be written as

$$T_{\varphi,b}f(x) = b(x)T_\varphi f(x) - T_\varphi(bf)(x).$$

For $b\in\mathrm{BMO}(w)$, under the same conditions as that in [1], Segovia and Torrea[3] have established the boundedness of commutator $T_{\varphi,b}$ from $L^p(w_1)$ into $L^p(w_2)$ with $1<p<\infty$, $w_1, w_2\in A_p$ and $w^p=\dfrac{w_1}{w_2}$.

To investigate the local behavior of solutions to the second order elliptic partial differential equations, Morrey[4] first introduced the classical Morrey space $M_{p,q}(\mathbf{R}^n)$ with the norm

$$\|f\|_{M_{p,q}(\mathbf{R}^n)} = \sup_{B\subset\mathbf{R}^n}\left(\frac{1}{|B|^{1-\frac{p}{q}}}\int_B |f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}},$$

where $f\in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ and $1\le p\le q<\infty$.

For some earlier work on $M_{p,q}(\mathbf{R}^n)$, see, e.g., [5–6]. For a recent account of the theory on the general case of $M_{p,q}(\mathbf{R}^n)$, we refer the reader to [7–9]. $M_{p,q}(\mathbf{R}^n)$ is a natural expansion of $L^p(\mathbf{R}^n)$ in the sense that $M_{p,p}(\mathbf{R}^n)=L^p(\mathbf{R}^n)$.

Komori and Shirai[10] introduced the weighted Morrey space, which is a natural generalization of the weighted Lebesgue space. Let $1\le p<q<\infty$ and w_1, w_2 be two functions. Then the norm of the weighted Morrey space $M_{p,q}(w_1, w_2)$ is defined by

$$\|f\|_{M_{p,q}(w_1,\, w_2)} = \sup_{B\subset\mathbf{R}^n}\left(\frac{1}{(w_2(B))^{\frac{1-p}{q}}}\int_B |f(x)|^p w_1(x)\mathrm{d}x\right)^{\frac{1}{p}}<\infty.$$

If $w_1=w_2=w$, we denote $M_{p,q}(w_1,\, w_2)=M_{p,q}(w)$. It is obvious that $M_{p,0}(w)=L^p(w)$ and $M_{p,1}(w)=L^\infty(w)$.

Inspired by [3, 10], we establish the weighted estimates for $T_{\varphi,b}$ on $M_{p,k}(w)$.