Communications
in
Mathematical
Research
31(4)(2015), 320–332

Biquartic Finite Volume Element Method Based on Lobatto-Guass Structure

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Communicated by Ma Fu-ming

Abstract: In this paper, a biquartic finite volume element method based on Lobatto-Guass structure is presented for variable coefficient elliptic equation on rectangular partition. Not only the optimal H^1 and L^2 error estimates but also some superconvergent properties are available and could be proved for this method. The numerical results obtained by this finite volume element scheme confirm the validity of the theoretical analysis and the effectiveness of this method.

Key words: Lobatto-Guass structure, biquartic, finite volume element method, error estimate, superconvergence

2010 MR subject classification: 65M15

Document code: A

Article ID: 1674-5647(2015)04-0320-13

DOI: 10.13447/j.1674-5647.2015.04.04

1 Introduction

Finite volume element methods (FVEMs) (see [1–2]), also called generalized difference methods (see [3–6]), have been widely used in numerical partial differential equations and achieved great development, due to the local conservation property and other attractive properties such as flexibility in handling complicated domain geometries and boundary conditions. In essence, FVEMs and finite element methods (FEMs) both are methods based on interpolation. Since the finite volume element methods were proposed, it has been found that some properties valid for the finite element methods (see [7–8]) are naturally valid for the finite volume element methods (see [3–6, 9–12]).

The systematic theoretical analysis for FEMs with Lobatto-Guass structure has been attached much attention. Chen[7,13–14] discussed the super-convergence of the numerical solution and the numerical gradient for the one-dimensional two-point boundary problem.

Received date: April 29, 2014.
Foundation item: The NSF (11371170) of China.
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When the finite element space takes order k, the numerical solutions and the numerical gradients for FEMs would have super-convergence at the zero points of Lobatto polynomials with order $k+1$ and at the k-th Guass points, respectively. In [7], these results are extended to higher dimensional elliptic problems for both rectangular and triangular meshes. By using unit orthogonal analysis, Chen[7] has proved all of the properties mentioned above. Motivated by the ideas of FEMs, people begin to apply the Lobatto-Guass structure to construct and discuss FVEMs in recent years. The works [16–18], respectively, analyze the first, the second and the third order FVEMs with Lobatto-Guass structure for one-dimensional two-points boundary problem. Therein not only the optimal H^1-norm and L^2-norm error estimates could be proved theoretically, but also the numerical solutions and the numerical gradients would have superconvergence at the primal partition's vertices and at dual partition's vertices, respectively. All above properties could be inherited for general optimal stress structure in [18]. For the two-dimensional elliptic equations, the superconvergent biquadratic finite volume element method on rectangular meshes is discussed in [19]. Ciarlet[20] generalized the schemes mentioned in [19] to quadrilateral meshes, and obtained the optimal L^2-norm error estimates.

If using the zeros of Lobatto polynomials to construct the Lagrange interpolation, then the corresponding optimal stress points (see [7, 17]) happen to be the Guass points. In this paper, we construct a biquartic finite volume element schemes on rectangular mesh by choosing the zeros of fifth-order Lobatto polynomials as primal partition's vertices and restricting the forth-order Guass points as the vertices of control volume, for two-order variable coefficient elliptic problems. The finite volume element schemes constructed by this way is proved to not only have $o(h^4)$ and $o(h^5)$ accuracy in H^1-norm and L^2-norm, respectively, but also have superconvergence for numerical gradients at the Guass points. In addition, we obtain a superconvergence for numerical solutions at the primal partition's vertices by numerical examples.

Let Q_hu be the Lagrange interpolating polynomial over interval $[-h,h]$ associated with five nodes $-h$, $-\sqrt{\frac{3}{7}}h$, 0, $\sqrt{\frac{3}{7}}h$, h, which are the null points of fifth-order Lobatto polynomial. Then

$$\begin{aligned}Q_hu =& \frac{7}{8}\Big(\xi^2-\frac{3}{7}\Big)\xi(\xi-1)u(-h)-\frac{49}{24}(\xi^2-1)\xi\Big(\xi-\sqrt{\frac{3}{7}}\Big)u\Big(-\sqrt{\frac{3}{7}}h\Big)\\&+\frac{7}{3}(\xi^2-1)\Big(\xi^2-\frac{3}{7}\Big)u(0)-\frac{49}{24}(\xi^2-1)\xi\Big(\xi+\sqrt{\frac{3}{7}}\Big)u\Big(\sqrt{\frac{3}{7}}h\Big)\\&+\frac{7}{8}\Big(\xi^2-\frac{3}{7}\Big)\xi(\xi+1)u(h),\end{aligned}\tag{1.1}$$

where $\xi=\frac{x}{h}$. The derivative of Q_hu at $x_0\in[-h,h]$ is

$$\begin{aligned}(Q_hu)'(x_0) =& \frac{1}{h^4}\Big[\Big(\frac{7}{2}x_0^3-\frac{21}{8}x_0^2h-\frac{3}{4}xh^2+\frac{3}{8}h^3\Big)u(-h)\\&+\Big(\frac{-49}{6}x_0^3+\frac{7}{8}x_0^2\sqrt{21}h+\frac{49}{12}x_0h^2-\frac{7}{24}\sqrt{21}h^3\Big)u\Big(-\sqrt{\frac{3}{7}}h\Big)\Big]\\&+\Big(\frac{28}{3}x_0^3-\frac{20}{3}x_0h^2\Big)u(0)\end{aligned}$$