CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0061

Vol. **x**, No. x, pp. 1-24
xx 2025

Optical Wave Turbulence: The Conformal Symmetry Transformations of Statistics of the Quantum Fluids of Light

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Received 6 December 2024; Accepted 22 April 2025

Abstract. We use the quantum fluid analogy [Nazarenko *et al.*, Phys. Rev. E, 92, 2015] for coherent structures (vortices or solitons) interacting among themselves and with the random wave component. This is performed for $2D$ defocusing media based on quantum fluid approximation of the two-dimensional nonlinear Schrödinger equation in the statistical frame. With this, the Lundgren-Monin-Novikov infinite chain of equations for the n-point density function f_n for the vorticity field is used. The conformal group of symmetry transformations calculated [Grebenev *et. al.*, Theor. Math. Phys., 217(2), 2023] is applied to implement several elements of a gauge theory in the conformal transformation optics. Finally, we demonstrate how to use the variational generalized Brenier principle [Brenier, J. Am. Math. Soc., 2, 1989] together with the conformal invariance of statistics to close the infinite chain of Landgren-Monin-Novikov equations.

AMS subject classifications: 57Z05, 57S25, 76F02

Key words: Optical wave turbulence, n-point statistics of a vortex field, gauge transformations, conformal invariance, generalized Brenier variational functional.

1 Introduction

We study an exciting emerging area: turbulence in nonlinear optical systems [8, 30–32]. It deals with such universal features of turbulence as the invariance of statistics of vorticity field shared by a great variety of applications, ranging from quantum to classical. The distinct property typical for quantum fluids of light is a tangle of randomly moving quantized vortex lines. Nonlinear optics is a hugely broad subject, covering many optical applications and a variety of optical media. For instance, it concerns with nonlinear optical fibres, liquid crystals, photo-refractive crystals, and light propagation through atomic

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vapours [25, 26]. Moreover, there are close analogies of optical processes with hydrodynamic turbulence in classical and condensed matter fluids such as superfluids [5]. For example, in both optics and condensed matter fluids, there exist vortex-like structures, shock waves, and weakly interacting random waves whose dynamics and statistics have similarities to random waves that appear on the water surface [5]. Optical systems, where a laser-produced light propagates in a non-linear medium, have one distinct feature in common – the coherence of electromagnetic waves [20]. Coherence is commonly considered the opposite to randomness or turbulence, which is why, perhaps, chaotic behavior was traditionally considered undesirable in such experiments until recently. The situation started to change drastically when it was realized that non-linear light exhibits interesting fluid-like behaviors with vortex structures and waves involved in complicated random motions having all essential features of classical turbulence [7].

At most fundamental perspective, the turbulence theory aims to understand the complex random interactions between turbulent fluctuations in systems containing a vast number of degrees of freedom. Turbulence is ubiquitous in Nature, it is observed in both the classical hydrodynamics, astrophysics and magneto-hydrodynamics, superfluids and Bose-Einstein condensate (BEC), weakly interacting waves, and nonlinear optics. Turbulence is one of the most important unsolved problems in physics, and developing a realistic description of the dynamics and statistics of its constituent entities, such as waves, vortices and other type of coherent structures, the transitions to turbulence, and the influence of turbulence on the other processes in natural and technological conditions would enable us to understand and control this important phenomenon which would bring about significant benefits to industry. It is natural to study nonlinear optics systems together because, at the most basic level, it share a nonlinear model based on the nonlinear Schrödinger equation (NSLE). Often the NSLE model is modified by adding a trapping potential term, terms accounting for dissipation, nonlocality of interaction, finite relaxation time, etc.

Another strong motivation for studying turbulent optical systems and designing respective experiments arises from strong links with a new area of non-equilibrium statistical mechanics – the distribution of n-point probability density functions (PDFs) f_n for the vorticity (scalar) fields, which is defined as a system of random vortexes with a broad frequency spectrum, which are involved in statistical nonlinear interactions. This is described by the Navier-Stokes equations in the statistical frame, i.e. the Lundgren-Monin-Novikov hierarchy [12] for f_n using the hydrodynamic analogy for NSLE. Simulations of $2D$ defocusing optical turbulence have been performed in various settings using the 2D defocusing NSLE model, i.e. with and without initial condensate present, forcing at large or at small scales. Simulations without an initial condensate and forcing at small scales exhibited presence of an inverse cascade in agreement with Wave Turbulence theory [21, Chapter 8.2, Remarks 8.2.2 and 8.2.3] predictions. Such a strongly nonequilibrium vortex condensation eventually lead to a strongly turbulent stage with interacting and annihilating vortices. It was noted that presence of an acoustic component facilitates the vortex annihilations, in agreement with a scenario which says that rapid cooling (damping acoustic phonons) leads to an arrest of the vortex annihilations and to persistence of vortices. Vortex annihilation is a process competing with the vortex clustering – a process well-known for the gas of point vortices in an ideal classical 2D fluid.

This paper aims at the interpretation of the conformal symmetry transformations constructed in [13] for the n-point statistics admitted by the hydrodynamic approximation of NLSE in 2D defocusing media. With this, we implement several elements of a gauge theory to the optical turbulence.

In nonlinear optics, understanding the conformal symmetry properties of light and, in particular, turbulence can lead to advances in technology related with light manipulations. It was reported in [35] the experimental observation of a photonic toroidal vortex as a new solution to the linear Schrödinger equation (the parabolic approximation), generated by the use of conformal mapping. The equation is considered in the cylindrical geometry with anomalous dispersion of the group velocity of the wave packet Ψ, which is invariant under conformal transformations of the plane at which two phase elements of Ψ are located. Lines of the cylindrical vortex tube are conformally transformed into circles with the formation of a vortex ring in three-dimensional space. The discussed equation can be used only in certain approximations disregarding nonlinear effects of the optical wave propagation and the interaction with the background of random waves. The justification of such structures should be considered in the framework of statistical symmetries of the vorticity field. When deriving the kinetic equation for the PDFs, we used in [13] the hydrodynamic NLSE approximation for the weight velocity field $\boldsymbol{u}$, as in [7].

In hydrodynamic turbulence, the numerical experiments performed by Bernard *et al.* [3,4] demonstrate that the zero-vorticity isolines for the two dimensional Euler equation with an external force and a uniform friction belong to the class of random curves that can be conformally mapped into a one-dimensional Brownian walk called Schramm-Löwner evolution (SLE_κ) curves [27]. The diffusion coefficient κ classifies the conformal invariant random curves. The SLE_6 with $\kappa=6$ first appears in the classical model of critical percolation and refers to a self-avoiding random walk [36]. The second example concerns to statistical properties of turbulent inverse cascades in a class of models devoted to a scalar field transported by two-dimensional flow such as surface quasi-geostrophic turbulence which describes a rotating stably stratified fluid, the asymptotic case of the Hasegawa-Mima equation for drift waves in magnetized plasma [18], the Charney and Oboukhov equation for waves in rotating fluids [19]. It was numerically demonstrated (see [3,4,9,10]) that the zero-isolines of the scalar field are statistically equivalent to conformal invariant curves and the zero-temperature isolines in surface quasi geostrophic (SQG) model belong to the universality class SLE_4 at large scales, i.e. for the inverse cascade.

The extension of symmetry to conformal invariance is the program proposed by Polyakov [23] for the two-dimensional statistical theory of turbulence. In such a case, the conformal group is infinite-dimensional, which allows to apply the methods of conformal field theory. For the two-dimensional turbulence, an analytic result about the conformal invariance of the one-point statistics of zero-vorticity isolines or the level set $\{\boldsymbol{x}\in R^2 : \boldsymbol{\omega}(\boldsymbol{x}) = 0\}$ or the contour of vortex clusters under the external forcing in the form of white Gaussian noise and the large-scale friction was obtained in [34] (see, also [16, 17, 33]). We performed a Lie group analysis of the first equation from the infinite Lundgren-Monin-Novikov (LMN) hierarchy [12] for the one-point probability density function f_1 (PDF) of vorticity. We proved that the conformal invariance is broken apart from points $\boldsymbol{x}\in R^2$ with zero-vorticity. The group G acts conformally with respect to the spatial variable $\boldsymbol{x}_{(1)}$ and transforms invariantly only "a fragment" of the first LMN equation, i.e. the $f_1(\boldsymbol{x}_{(1)},\boldsymbol{\omega}_{(1)})|_{\boldsymbol{\omega}_{(1)}=0}$. The conformal invariance of the normalization and reduction properties, the separation and coincidence properties [12] of the PDF's were also proven.

The above-mentioned findings were expanded in [13,14] to the case of an arbitrary f_n. The conceptual novelty of the work [13] consists in the representation of the group $G = G_1\times\cdots\times G_n$ as a fiber bundle $\mathcal{P} = P_{\boldsymbol{x}_{(1)}}\times\cdots\times P_{\boldsymbol{x}_{(n)}}$ over the flow space X (the configuration space of a $2D$ turbulent flow) with the base $\boldsymbol{x}_{(j)}$, where the fiber $P_{\boldsymbol{x}_{(j)}}$ is a group of transformation G_j, which allows defining the gauge transformations of a fiber. In the differential geometry such fiber are called principal. This refers to the well-knows geometric options of Yang-Mills fields of the gauge transformations [2]. Here the set $(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)})\in X$ is the n-point sample in the statistical notion to which the observable data $(\omega_{(1)},\ldots,\omega_{(n)})$, (the value of the vorticity component $\omega(\boldsymbol{x}_{(i)})$) at the point $\boldsymbol{x}_{(i)}$, correspond.

In Section 2, we introduce the basic notions of statistical turbulence and shortly review the results obtained in [13]. The hydrodynamic NLSE approximation defined by the Euler equation will be presented according to the work [7]. The equation for the n-point statistic f_n of the vortex field $\boldsymbol{\omega}$ are derived from an infinite chain of LMN equations (statistical form of the Euler equations). The Gaussian white-in-time forcing and large-scale friction together with the viscous term are implementing to the f_n-equation to generate both the large and small scales of turbulent motions. In the following Section 4, we delivery an infinite-dimensional Lie algebra $\mathfrak{g}$ of the Lie group G symmetry transformations of the f_n-equation of LMN chain. The basis elements of $\mathfrak{g}$ will be constructed and the structure of this infinite-dimensional Lie algebra is given. With this, we present the gage potential and curvature tensor. The holomorphic bundle over the flow space X will be also discussed. Finally, in Section 5 we show how the variational generalized Brenier principle can be used to close the LMN infinite chain. A discussion and summary of results are given in Section 6.

2 Basic notions, hydrodynamic NLSE approximation

2.1 Statistical concept

The n-point sample $(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)})\in X$ is considered as a set of labeled points of the flow domain X to which the observable data $(\omega_{(1)},\ldots,\omega_{(n)})$ (the value of the vorticity $\omega(\boldsymbol{x}_{(i)})$ which is a scalar field in 2D) at the point $\boldsymbol{x}_{(i)}$ correspond. The brackets for the indexed variable $\boldsymbol{x}_{(i)}$ will refer to denote the sample variable. The set of states $(\omega_{(1)},\ldots,\omega_{(n)})$ of the n-point sample $(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)})$ for turbulent flows is the direct product $\mathcal{M}^n = \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \simeq R^n$. The space of states of a point $\boldsymbol{x}_{(i)}$ of the turbulent flow is a one-dimensional bundle $\mathcal{M}_i \simeq R$ over X.

We consider the family of probabilistic measures $\{\mu_{\boldsymbol{x}_{(i)}} | \boldsymbol{x}_{(i)} \in X\}$ defined on $\mathcal{M}_i$ parametrized by $\boldsymbol{x}_{(i)}$ and the standard Lebesgue measure ν on $\mathcal{M}_i$. It is assumed that the family of measures $\mu_{\boldsymbol{x}_{(i)}}$ is absolutely continuous with respect to the measure ν defined on $\mathcal{M}_i \simeq R$. Then the map

$$\mu_{\boldsymbol{x}_{(i)}} \rightarrow \left[\frac{\mathrm{d}\mu_{\boldsymbol{x}_{(i)}}}{\mathrm{d}\nu}\right]^{1/2}, \tag{2.1}$$

where $\mathrm{d}\mu_{\boldsymbol{x}_{(i)}}/\mathrm{d}\nu$ is the Radon-Nikodym derivative, defines an embedding of the family of probabilistic measures $\{\mu_{\boldsymbol{x}_{(i)}} | \boldsymbol{x}_{(i)} \in X\} \in L^2(\mathcal{M}_i,\nu)$ into the unit sphere S of the Banach space $L^1(\mathcal{M}_i,\nu)$. The derivative

$$\frac{\mathrm{d}\mu_{\boldsymbol{x}_{(i)}}}{\mathrm{d}\nu} = f \tag{2.2}$$

defines the PDF in the Banach space S. The concrete form of the PDF will be given below for the model of wave optical turbulence.

2.2 Optical turbulence: Nonlinear Shrödinger equation

An analogy between the behavior of optical and hydrodynamic fields is applied for the statistical representation of the defocusing NLSE. Nonlinear propagation of optical waves in terms of a scalar wave complex function $\psi(x,y,t)$ for envelopes can be described by the NLSE, which in terms of the dimensionless variable x,y,t and ψ reads

$$i\psi_t + \Delta\psi + \psi - |\psi|^2\psi = 0. \tag{2.3}$$

The Madelung transformation

$$\psi = \sqrt{\rho}e^{i\phi}, \quad |\psi|^2 = \rho, \tag{2.4}$$

where $|\psi|^2$ denotes the optical intensity and ϕ is the wave function phase, establishes the correspondence between the optical and hydrodynamic field: ρ and $\boldsymbol{v} = \nabla\phi$ satisfy the Euler equations for the nonviscous polytropic gas with the adiabatic exponent $\gamma = 2$, the nonlinear perturbation of the refractive index corresponds to the pressure p and the traveled distance of the optical wave is associated with the time t. At the points $\mathbf{x_i} = (x_i,y_i) \in R^2$, where $\psi = 0$ the phase ϕ is undefined, and the vorticity is defined by the delta function $\delta(\mathbf{x} - \mathbf{x_i})$. The transition to the velocity field $\boldsymbol{u} = \sqrt{\rho}\boldsymbol{v}$ [7] and using the approximation of $\boldsymbol{v}$ at the vorticity center $\mathbf{x_i}$ the Pitaevskii solution leads to the vorticity field which is still localized and rapidly decreasing at large distances from the vortex center but is no longer a delta-function distribution. The NLSE Hamiltonian

$$H = \int \left[|\nabla\psi|^2 + \frac{1}{2}(|\nabla\psi|^2 - 1)^2\right] d\mathbf{x} \tag{2.5}$$

in terms of the hydrodynamical variables $\boldsymbol{u},\rho$ takes the form

$$H=H_K+H_0,\quad H_K=\frac{1}{2}\int u^2 d\mathbf{x},\quad H_0=\frac{1}{2}\int[(\rho-1)^2+2|\nabla\sqrt{\rho}|^2]d\mathbf{x},\quad u=|\boldsymbol{u}|. \tag{2.6}$$

The first term H_K coincides with the Hamiltonian of an ideal incompressible fluid and H_K is dominant on motion scales of the order of the vortex core D radius ξ [7]. Moreover, the divergence of the field $\boldsymbol{u}$, i.e. $\gamma(\mathbf{x})=\nabla\cdot\boldsymbol{u}$ is $\gamma(\mathbf{x})\ll 1$ [7] on these motion scales. Thus, on the motion scales of the order of ξ, the hydrodynamic NLSE approximation is defined by the Euler equation of the ideal fluid. The vorticity $\Omega=\omega\mathbf{e}_z$ and velocity field $\boldsymbol{u}$ defined on a neighborhood of the point $\boldsymbol{x}$ is defined by the Biot-Savart law

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{2\pi}\int_D \mathrm{d}\boldsymbol{x}'\omega(\boldsymbol{x}',t)\mathbf{e}_z\times\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^2}=\int_D \mathrm{d}\boldsymbol{x}'\boldsymbol{U}(\boldsymbol{x}-\boldsymbol{x}',t)\omega(\boldsymbol{x}',t), \tag{2.7}$$

where $\boldsymbol{U}$ is the vortex velocity with an integrable singularity and D is a domain where the vorticity field is defined.

2.3 Statistical representation of the field of optical vortices

With the hydrodynamic NLSE approximation defined by the Euler equation for the ideal fluid is known, the equation for the n-point PDF f_n of the vortex field can be determined from the infinite chain of LMN equations, which is derived from the statistical form of the Euler equations by the Biot-Savart law. The f_n-equation is considered under the external action of white-Gaussian noise and the large-scale Ekman friction, which makes the PDF statistically stationarity

$$\frac{\partial f_n}{\partial t}=-\sum_{j=1}^{n}\left[\frac{\partial}{\partial x_{(j)}}\langle u(\boldsymbol{x}_{(j)},t)|\{\omega_l,\boldsymbol{x}_{(l)}\}\rangle f_n+\frac{\partial}{\partial y_{(j)}}\langle v(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle f_n\right]=\mathcal{F}, \tag{2.8}$$

where $n=1,\dots,\infty$. The velocity components are given by the formulas using the Biot–Savart law for vorticity fields defined on the vortex core D_j of radius ξ_j centred at $\boldsymbol{x}_j$ wherein the velocity is calculated

$$\begin{aligned}&\langle u(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle \\ &=\int_{D_j}\int_{-\infty}^{\infty}\mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^1(\boldsymbol{x}_{(l)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}(\boldsymbol{x}_{(n+1)},\omega_{(n+1)},\{\boldsymbol{x}_{(l)},\omega_{(l)}\},t)}{f_n(\{\boldsymbol{x}_{(l)},\omega_{(l)}\},t)},\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\langle v(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle \\ &=\int_{D_j}\int_{-\infty}^{\infty}\mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^2(\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}(\boldsymbol{x}_{(n+1)},\omega_{(n+1)},\{\boldsymbol{x}_{(l)},\omega_{(l)}\},t)}{f_n(\{\boldsymbol{x}_{(l)},\omega_{(l)}\},t)},\end{aligned} \tag{2.10}$$

where

$$\boldsymbol{r}_{(sd)}=\boldsymbol{x}_{(s)}-\boldsymbol{x}_{(d)},\quad \alpha^1(\boldsymbol{r}_{(s,d)})=-\frac{1}{2\pi}\frac{r^2_{(s,d)}}{|\boldsymbol{r}_{(s,d)}|^2},\quad \alpha^2(\boldsymbol{r}_{(s,d)})=\frac{1}{2\pi}\frac{r^1_{(s,d)}}{|\boldsymbol{r}_{(s,d)}|^2}. \tag{2.11}$$

The integrations in (2.9) and (2.10) with respect to the variable $\boldsymbol{x}_{(n+1)}$ are taken over the vortex core D_j with $\boldsymbol{x}_{(n+1)} \neq \boldsymbol{x}_j$. The right-hand side of the equation reads

$$\mathcal{F} = \beta \frac{\partial}{\partial \omega_{(n)}} (\omega_{(n)} f_n) - \frac{1}{2} \sum_{j=1}^{n} Q(x_{(n)} - x_{(j)}) \frac{\partial^2}{\partial \omega_{(j)}^2} f_n. \tag{2.12}$$

The first term in (2.12) is the friction damping (the Ekman friction), by means of which the interaction energy is transferred toward large scales of the inverse cascade. The second term corresponds to exciting the system by white-Gaussian noise with a short correlation radius, and $Q(x_{(n)} - x_{(j)})$ is an external action amplitude.

The class of PDFs is defined by the relationships

$$\int d\omega_{(1)} \ldots d\omega_{(n)} f_n = 1, \quad \int d\omega_{(n+1)} f_{n+1} = f_n, \tag{2.13}$$

$$\begin{aligned} &\lim_{|\boldsymbol{x}_{(n)} - \boldsymbol{x}_{(n+1)}| \to \infty} f_{n+1}(\boldsymbol{x}_{(1)}, \omega_{(1)}, \ldots, \boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, t) \\ &= f_1(\boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, t) \cdot f_n(\boldsymbol{x}_{(1)}, \omega_{(1)}, \ldots, \boldsymbol{x}_{(n)}, \omega_{(n)}, t), \end{aligned} \tag{2.14}$$

$$\lim_{|\boldsymbol{x}_{(n)} - \boldsymbol{x}_{(n+1)}| \to 0} f_{n+1} = \delta(\omega_{(n+1)} - \omega_{(n)}) f_n. \tag{2.15}$$

3 Symmetry transformations of the n-point PDFs

We briefly review the results presented in [13] about the symmetry transformations of the f_n-equation of LMN chain.

3.1 Infinitesimal operator and gauge transformations

Let us consider an arbitrary $\boldsymbol{x}_{(j)}$ of the n-point sample and a fiber $P_{\boldsymbol{x}_{(j)}}$ over the basis point $\boldsymbol{x}_{(j)} \in X$ which is defined by the infinitesimal operator $S_{(j)}$ or the so-called fundamental vector field tangents to $P_{\boldsymbol{x}_{(j)}}$

$$\begin{aligned} S_{(j)} = &\, \xi^1 \frac{\partial}{\partial x_1^1} + \xi^2 \frac{\partial}{\partial x_1^2} + \xi^3 \frac{\partial}{\partial \omega_{(1)}} + \cdots \\ &+ \xi^{3n-2} \frac{\partial}{\partial x_n^1} + \xi^{3n-1} \frac{\partial}{\partial x_n^2} + \xi^{3n} \frac{\partial}{\partial \omega_{(n)}} + \eta^1_{(n)} \frac{\partial}{\partial f_n} \\ &+ \xi^{3n+1} \frac{\partial}{\partial x_{n+1}^1} + \xi^{3n+2} \frac{\partial}{\partial x_{n+1}^2} + \xi^{3n+3} \frac{\partial}{\partial \omega_{(n+1)}} + \eta^2_{(n)} \frac{\partial}{\partial f_{n+1}}, \end{aligned} \tag{3.1}$$

where j indexes $\xi^{3n+1}, \xi^{3n+2}, \xi^{3n+3}$. Here $\boldsymbol{x}_k$ refers to vectors of the vector space R^2. The coordinates of the infinitesimal operator are defined by the formulas

$$\xi^1 = c^{11}(\boldsymbol{x}_1) x_1^1 + c^{12}(\boldsymbol{x}_1) x_1^2 + d^1(\boldsymbol{x}_1), \tag{3.2}$$

$$\xi^2 = c^{21}(\boldsymbol{x}_1) x_1^1 + c^{22}(\boldsymbol{x}_1) x_1^2 + d^2(\boldsymbol{x}_1), \tag{3.3}$$

$$\xi^3 = \big[6c^{11}(\boldsymbol{x}_1)\big]\omega_{(1)}, \tag{3.4}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\xi^{3k-2} = c^{11}(\boldsymbol{x}_k)x_k^1 + c^{12}(\boldsymbol{x}_k)x_k^2 + d^1(\boldsymbol{x}_k), \tag{3.5}$$

$$\xi^{3k-1} = c^{21}(\boldsymbol{x}_k)x_k^1 + c^{22}(\boldsymbol{x}_k)x_k^2 + d^2(\boldsymbol{x}_k), \tag{3.6}$$

$$\xi^{3k} = \big[6c^{11}(\boldsymbol{x}_k)\big]\omega_{(k)}, \tag{3.7}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\xi^{3n-2} = c^{11}(\boldsymbol{x}_n)x_n^1 + c^{12}(\boldsymbol{x}_n)x_n^2 + d^1(\boldsymbol{x}_n), \tag{3.8}$$

$$\xi^{3n-1} = c^{21}(\boldsymbol{x}_n)x_n^1 + c^{22}(\boldsymbol{x}_n)x_n^2 + d^2(\boldsymbol{x}_n), \tag{3.9}$$

$$\xi^{3n} = \big[6c^{11}(\boldsymbol{x}_{(n)})\big]\omega_{(n)}, \tag{3.10}$$

$$\xi^{3n+1} = c^{11}(\boldsymbol{x}_j)x_{n+1}^1 + c^{12}(\boldsymbol{x}_j)x_{n+1}^2 + d^1(\boldsymbol{x}_j), \tag{3.11}$$

$$\xi^{3n+2} = c^{21}(\boldsymbol{x}_j)x_{n+1}^1 + c^{22}(\boldsymbol{x}_j)x_{n+1}^2 + d^2(\boldsymbol{x}_j), \tag{3.12}$$

$$\xi^{3n+3} = \big[2c^{11}(\boldsymbol{x}_j)\big]\omega_{(n+1)}, \tag{3.13}$$

where $k=1,\ldots,n$, the coefficients c^{ls} satisfy the equalities $c^{11}=c^{22}, c^{12}=-c^{21}, c^{11}, c^{12}$ are arbitrary harmonic functions. The functions $d^1(\boldsymbol{y})$ and $d^2(\boldsymbol{y})$ are of the form

$$d_1^1(\boldsymbol{y}) = 2c^{11}(\boldsymbol{y}) - c_1^{11}(\boldsymbol{y})y^1 - c_1^{12}(\boldsymbol{y})y^2, \tag{3.14}$$

$$d_2^1(\boldsymbol{y}) = -c_2^{11}(\boldsymbol{y})y^1 - c_2^{12}(\boldsymbol{y})y^2, \tag{3.15}$$

$$d_1^2(\boldsymbol{y}) = c_1^{12}(\boldsymbol{y})y^1 - c_1^{11}(\boldsymbol{y})y^2, \tag{3.16}$$

$$d_2^2(\boldsymbol{y}) = 2c^{11}(\boldsymbol{y}) + c_2^{12}(\boldsymbol{y})y^1 - c_2^{22}(\boldsymbol{y})y^2. \tag{3.17}$$

The coordinates $\eta^1_{(n)}$ and $\eta^2_{(n)}$ have the form

$$\eta^1_{(n)} = a^{00}_{(n)}(t,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)f_n, \tag{3.18}$$

$$a^{00}_n = \frac{\partial\xi^0}{\partial t} - \left(\frac{\partial\xi^0}{\partial t} + \frac{\partial\xi^1}{\partial x_1^1} + \frac{\partial\xi^2}{\partial x_1^2} + \cdots + \frac{\partial\xi^{3n-2}}{\partial x_n^1} + \frac{\partial\xi^{3n-1}}{\partial x_n^2}\right),$$

$$\eta^2_{(n)} = a^{00}_{(n+1)}(t,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{n+1})f_{(n+1)}, \tag{3.19}$$

$$a^{00}_{(n+1)} = \frac{\partial\xi^0}{\partial t} - \left(\frac{\partial\xi^0}{\partial t} + \frac{\partial\xi^1}{\partial x_1^1} + \frac{\partial\xi^2}{\partial x_1^2} + \frac{\partial\xi^4}{\partial x_2^1} + \frac{\partial\xi^5}{\partial x_2^2} + \cdots + \frac{\partial\xi^{3n+1}}{\partial x_{n+1}^1} + \frac{\partial\xi^{3n+2}}{\partial x_{n+1}^2}\right). \tag{3.20}$$

The functions ξ^{3k-1}, ξ^{3k-2} satisfy the Cauchy-Riemann conditions, i.e. these are analytical functions. Using $\xi^1, \xi^2, \ldots, \xi^{3n-2}, \xi^{3n-1}, \xi^{3n}$ and ξ^{3n+1}, we obtain

$$\eta^1_{(n)} = -6\sum_{i=1}^{n} c^{11}(\boldsymbol{x}_i), \tag{3.21}$$

$$\eta^2_{(n)} = -2c^{11}(\boldsymbol{x}_j) - 6\sum_{i=1}^{n} c^{11}(\boldsymbol{x}_i). \tag{3.22}$$

Using $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_n$, we introduce the complex variables

$$z_1 = x_1^1 + i x_1^2, \quad z_2 = x_2^1 + i x_2^2, \quad \ldots, \quad z_n = x_n^1 + i x_n^2, \tag{3.23}$$

or

$$z_1 = x_1 + i y_1, \quad z_2 = x_2 + i y_2, \quad \ldots, \quad z_n = x_n + i y_n. \tag{3.24}$$

The infinitesimal operator $S_{(j)}$ generates a Lie group G_j (the group parameter a is omitted from the notation)

$$z_1^* = F(z_1), \tag{3.25}$$
$$\omega_1^* = |F_{z_1}|^2 \omega_{(1)}, \tag{3.26}$$
$$\ldots \quad \ldots \quad \ldots$$
$$z_k^* = F(z_k), \tag{3.27}$$
$$\omega_{(k)}^* = |F_{z_k}|^2 \omega_{(k)}, \tag{3.28}$$
$$\ldots \quad \ldots \quad \ldots$$
$$z_{n+1}^* = F'(z_j, z_{n+1}) = F(z_j) + (z_{n+1} - z_j) \frac{\mathrm{d}F(z_j)}{\mathrm{d}z_j} \left| \frac{\mathrm{d}F(z_j)}{\mathrm{d}z_j} \right|^{-2/3}, \tag{3.29}$$
$$\omega_{(n+1)}^* = |F_{z_j}|^{2/3} \omega_{(n+1)}, \tag{3.30}$$
$$f_n^* = \prod_{k=1}^{n} |F_{z_k}|^{-2} f_n, \tag{3.31}$$
$$f_{n+1}^* = |F_{z_j}|^{-2/3} \prod_{k=1}^{n} |F_{z_k}|^{-2} f_{n+1}, \tag{3.32}$$

where $F = U + iV, U$ and V are arbitrary conjugate harmonic functions satisfying the Cauchy-Riemann conditions. The transformations of the variables $z_{n+1}, \omega_{(n+1)}$ and f_{n+1} are determined by the group G_j, i.e. depends on the index j.

We define

$$G = G_1 \times \cdots \times G_n \tag{3.33}$$

as a direct product of the Lie groups G_j, and G is again a Lie group acting in $\mathcal{P} = P_{\boldsymbol{x}_{(1)}} \times \cdots \times P_{\boldsymbol{x}_{(n)}}$. With this, a mapping

$$g : (\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(n)}) \;\mapsto\; G \tag{3.34}$$

is a gage transformation [2].

3.2 G-invariance of the f_n-equation

We consider the f_n-equation from the infinite chain of LMN equations and the n-point sample $(\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(n)}) \in X$ wherein the PDF f_n is defined. The specification of the right-hand side $\mathcal{F}$ by (2.12) leads to the stationary of Eq. 2.8

$$\sum_{j=1}^{n}\mathrm{Re}\big(\nabla_{z_{(j)}}\cdot\big[\langle\mathcal{U}(z_{(j)},\bar{z}_{(j)})\,|\,\{\omega_{(l)},z_{(l)},\bar{z}_{(l)}\}\rangle\big]\big)f_n=\mathcal{F}, \tag{3.35}$$

where $j=1,\ldots,n$, Re is the real part of the complex number. Eq. (3.35) defines the manifold $\mathcal{E}$ in the jet space J^2. The velocity components $\langle u(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle$, $\langle v(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle$ define the velocity vector in C^n with the component

$$\langle\mathcal{U}(z_{(j)},\bar{z}_{(j)},t)|\{\omega_{(l)},z_{(l)},\bar{z}_{(l)}\}\rangle=\langle u(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle+i\langle v(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle. \tag{3.36}$$

The velocity vector

$$\big(\langle\mathcal{U}(z_{(1)},\bar{z}_{(1)})|\{\omega_{(l)},z_{(l)},\bar{z}_{(l)}\}\rangle,\ldots,\langle\mathcal{U}(z_{(n)},\bar{z}_{(n)})\,|\,\{\omega_{(l)},z_{(l)},\bar{z}_{(l)}\}\rangle\big)\in C^n \tag{3.37}$$

is transformed under the action of the group G, substituting instead of $\boldsymbol{x}_k$ (z_k) the sample variable $\boldsymbol{x}_{(k)}$ ($z_{(k)}$) into the formulas (3.25)-(3.32), according to the following formulas [13]:

$$\langle u^*(\boldsymbol{x}^*_{(j)})|\omega^*_{(l)},\boldsymbol{x}^*_{(l)}\rangle=\left[\frac{\partial U}{\partial x_{(j)}}\langle u(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle+\frac{\partial U}{\partial y_{(j)}}\langle v(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle\right], \tag{3.38}$$

$$\langle v^*(\boldsymbol{x}^*_{(j)})|\omega^*_{(l)},\boldsymbol{x}^*_{(l)}\rangle=\left[-\frac{\partial U}{\partial y_{(j)}}\langle u(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle+\frac{\partial U}{\partial x_{(j)}}\langle v(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle\right]. \tag{3.39}$$

The transformed divergence reads

$$\begin{aligned}&\nabla^*_{\boldsymbol{x}^*_{(j)}}\cdot[\langle\mathcal{U}^*(\boldsymbol{x}^*_{(j)})|\omega^*_{(l)},\boldsymbol{x}^*_{(l)}\rangle f^*_n]\\&=\gamma\nabla_{\boldsymbol{x}_{(j)}}\cdot[\langle\mathcal{U}(\boldsymbol{x}_{(j)})|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle f_n]+\mathcal{G}(\boldsymbol{x}_{(j)},\omega_{(j)}),\end{aligned} \tag{3.40}$$

where $\gamma=\prod_{i=1}^{n}|F_{z_{(i)}}|^{-2}$,

$$\mathcal{G}(\boldsymbol{x}_{(j)},\omega_{(j)})=\frac{\omega_{(j)}}{\gamma}\left[\frac{\partial\gamma}{\partial x_{(j)}}\frac{\partial}{\partial\omega_{(j)}}\langle u(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle+\frac{\partial\gamma}{\partial y_{(j)}}\frac{\partial}{\partial\omega_{(j)}}\langle v(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle\right]. \tag{3.41}$$

Substituting the obtained expressions into the left-hand side of the equation, we find in terms of the transformed variables that in the statistical sample $\{\boldsymbol{x}_{(j)},0\}$ (the sample located on a zero-vorticity curve $\mathbf{x}(l)$) the value $\mathcal{G}(\boldsymbol{x}_{(j)},0)=0$. With this, we obtain

$$\begin{aligned}&\nabla^*_{\boldsymbol{x}^*_{(j)}}\cdot[\langle\mathcal{U}^*(\boldsymbol{x}^*_{(j)})\,|\,\omega^*_{(l)},\boldsymbol{x}^*_{(l)}\rangle f^*_n]\big|_{\omega^*_{(j)}=0}\\&=\gamma\nabla_{\boldsymbol{x}_{(j)}}\cdot[\langle\mathcal{U}(\boldsymbol{x}_{(j)})|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle f_n]\big|_{\omega_{(j)}=0}.\end{aligned} \tag{3.42}$$

We note that $\omega_{(j)}=0$ is transformed by the group G_j into $\omega^*_{(j)}=0$. Therefore, the left-hand side of Eq. (3.35) becomes

$$\begin{aligned}&\sum_{j=1}^{n}\mathrm{Re}\big(\nabla_{z^*_{(j)}}\cdot[\langle\mathcal{U}^*(z^*_{(j)},\bar{z}^*_{(j)})|\{\omega^*_{(l)},z^*_{(l)},\bar{z}^*_{(l)}\}\rangle]\big)f^*_n\big|_{\omega^*_{(j)}=0}\\&=\gamma\sum_{j=1}^{n}\mathrm{Re}\big(\nabla_{z_{(j)}}\cdot[\langle\mathcal{U}(z_{(j)},\bar{z}_{(j)})|\{\omega_{(l)},z_{(l)},\bar{z}_{(l)}\}\rangle]\big)f_n\big|_{\omega_{(j)}=0}.\end{aligned}\tag{3.43}$$

The conformal transformations F define a Lie pseudogroup of transformations X, i.e. they are defined locally. We consider the right-hand side of Eq. (3.35), the first term of $\mathcal{F}$ is transformed according to (see for details [13])

$$\beta\frac{\partial}{\partial\omega^*_{(n)}}(\omega^*_{(n)}f^*_n)=\gamma\beta\frac{\partial}{\partial\omega_{(n)}}(\omega_{(n)}f_n).\tag{3.44}$$

The second term in $\mathcal{F}$ then becomes

$$\frac{1}{2}\sum_{j=1}^{n}Q^*(x^*_{(n)}-x^*_{(j)})\frac{\partial^2}{\partial\omega^{*2}_{(j)}}f^*_n=\frac{1}{2}\sum_{j=1}^{n}Q^*(x^*_{(n)}-x^*_{(j)})|F_{z_{(j)}}|^{-4}\gamma\frac{\partial^2}{\partial\omega^2_{(j)}}f_n.\tag{3.45}$$

With this, the invariance of (3.35) requires a condition for the transformation of $Q(x_{(n)}-x_{(j)})$

$$Q^*(x^*_{(n)}-x^*_{(j)})=|F_{z_{(j)}}|^4Q(x_{(n)}-x_{(j)}).\tag{3.46}$$

Thus, if the condition (3.46) is satisfied then Eq. (3.35) is invariant transformed on the sample $(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)})\in X$ with $\omega_{(j)}=0, j=1,\ldots,n$. We present now the influence of viscosity on the symmetry transformations of Eq. (3.35). The viscosity transfers the turbulence energy toward small scales with the formation of the direct energy transfer. It is well known from numerical experiments the scale invariance is broken in almost all known direct cascades, see [9]. To take the viscosity into account, the right-hand side of (3.35) is supplemented with the term

$$\begin{aligned}\mathcal{K}&=\nu\sum_{j=1}^{n}\mathcal{K}_j\equiv\nu\sum_{j=1}^{n}\frac{\partial}{\partial\omega_{(j)}}\left(\int\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\int\mathrm{d}\boldsymbol{x}_{(n+1)}\delta(\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)})f_{n+1}\right)\\&=\nu\lim_{|\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)}|\to0}\sum_{j=1}^{n}\frac{\partial}{\partial\omega_{(j)}}\left(\int\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\Delta_{\boldsymbol{x}_{(n+1)}}f_{n+1}\right),\end{aligned}\tag{3.47}$$

where $\Delta_{\boldsymbol{x}_{(n+1)}}$ is the Laplacian in the variables $\boldsymbol{x}_{(n+1)}$. The calculations of $\mathcal{K}^*$ performed in [13] and $\mathcal{K}^*$ transforms in accordance with the formula

$$\mathcal{K}^*=\gamma\nu\sum_{j=1}^{n}|F_{z_{(j)}}|^{-4/3}\mathcal{K}_j.\tag{3.48}$$

Hence, viscosity is symmetry breaking with respect to the action of G.

4 Lie algebra of the G group

We present an infinite-dimensional Lie algebra $\mathfrak{g}$ of the Lie group G symmetry transformations. The basis elements of $\mathfrak{g}$ will be constructed and the structure of this infinite-dimensional Lie algebra is given. We present the gage potential and curvature tensor. The holomorphic bundle over the flow space X will be also discussed.

4.1 Witt-type algebra

The infinitesimal operator $S_{(j)}$ induces an infinite dimensional Lie algebra $\mathfrak{g}_j$ and the corresponding Lie group G_j is an infinite dimensional pseudo-group Lie. Using the operator $S_{(j)}$, we present the basis of $\mathfrak{g}_j$. First, we recast S_j in the complex variables frame, the infinitesimal operator $S_{(j)}$, as given in (3.1) has the form

$$
\begin{aligned}
S_{(j)} = & \psi^1 \frac{\mathrm{d}}{\mathrm{dz}_1} + \bar{\psi}^1 \frac{\mathrm{d}}{\mathrm{d\bar{z}}_1} + (\psi^1_{z_1} + \bar{\psi}^1_{\bar{z}_1})\omega_{(1)} \frac{\mathrm{d}}{\mathrm{d}\omega_{(1)}} + \cdots \\
& + \psi^n \frac{\mathrm{d}}{\mathrm{dz_n}} + \bar{\psi}^n \frac{\mathrm{d}}{\mathrm{d}\bar{z}_n} + (\psi^n_{z_n} + \bar{\psi}^n_{\bar{z}_n})\omega_{(n)} \frac{\mathrm{d}}{\mathrm{d}\omega_{(n)}} - \sum_{i=1}^{n} (\psi^i_{z_i} + \bar{\psi}^i_{\bar{z}_i}) f_n \frac{\mathrm{d}}{\mathrm{df_n}} \\
& + \psi' \frac{\mathrm{d}}{\mathrm{dz_{n+1}}} + \bar{\psi}' \frac{\mathrm{d}}{\mathrm{d\bar{z}_{n+1}}} + \frac{2}{3}(\psi'_{z_{n+1}} + \bar{\psi}'_{\bar{z}_{n+1}})\omega_{(n+1)} \frac{\mathrm{d}}{\mathrm{d}\omega_{(n+1)}} \\
& - \left(\frac{2}{3}(\psi'_{z_j} + \bar{\psi}'_{\bar{z}_j}) + \sum_{i=1}^{n} (\psi'_{z_i} + \bar{\psi}'_{\bar{z}_i}) \right) f_{n+1} \frac{\mathrm{d}}{\mathrm{df_{n+1}}},
\end{aligned} \tag{4.1}
$$

where $\psi^1 = \xi^1 + i\xi^2, \ldots, \psi' = \xi^{3n+1} + i\xi^{3n+2}$. Further, we will use the notations $z' = z_{n+1}$ and $\omega' = \omega_{(n+1)}$ The symbols $\psi^i_{z_i}$ $(\bar{\psi}^i_{\bar{z}_i})$ denote the derivative with respect to the variables z_i $(\bar{z}_i)$ respectively. To rewrite the operator $S_{(j)}$ in the complex variable frame, we used the formulas

$$
\frac{\partial}{\partial x} = \frac{\mathrm{d}}{\mathrm{dz}} + \frac{\mathrm{d}}{\mathrm{d\bar{z}}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\mathrm{d}}{\mathrm{dz}} - \frac{\mathrm{d}}{\mathrm{d\bar{z}}} \right). \tag{4.2}
$$

Then the basis elements of the Lie algebra generated by $S_{(j)}$ are of the form

$$
\begin{aligned}
k^m_j = & -\sum_{i=1}^{n} z_i^{m+1} \frac{\mathrm{d}}{\mathrm{dz_i}} - (m+1)\sum_{i=1}^{n} z_i^m \omega_{(i)} \frac{\mathrm{d}}{\mathrm{d!_{(i)}}} + (m+1)\sum_{i=1}^{n} z_i{}^m f_n \frac{\mathrm{d}}{\mathrm{df_n}} \\
& - z_j^{m+1}(z' + c_n) \frac{\mathrm{d}}{\mathrm{dz'}} + \frac{2}{3}(m+1) z_j^m \omega' \frac{\mathrm{d}}{\mathrm{d}\omega'} \\
& - \left(\frac{2}{3}(m+1) z_j^m + (m+1)\sum_{i=1}^{n} z_i^m \right) f_{n+1} \frac{\mathrm{d}}{\mathrm{df_{n+1}}},
\end{aligned} \tag{4.3}
$$

$$
\bar{k}^m_j = -\sum_{i=1}^{n} \bar{z}_i^{m+1} \frac{\mathrm{d}}{\mathrm{d\bar{z}_i}} - (m+1)\sum_{i=1}^{n} \bar{z}_i^m \omega_{(i)} \frac{\mathrm{d}}{\mathrm{d}\omega_{(i)}} + (m+1)\sum_{i=1}^{n} \bar{z}_i^m f_n \frac{\mathrm{d}}{\mathrm{df_n}}
$$

$$
\begin{aligned}
&-\bar{z}_j^{m+1}(\bar{z}'+c_n)\frac{\mathrm{d}}{\mathrm{d}\bar{z}'}+\frac{2}{3}(m+1)\bar{z}_j^m\omega'\frac{\mathrm{d}}{\mathrm{d}\omega'}\\
&-\left(\frac{2}{3}(m+1)\bar{z}_j^m+(m+1)\sum_{i=1}^n\bar{z}_i^m\right)f_{n+1}\frac{\mathrm{d}}{\mathrm{df_{n+1}}},
\end{aligned}
\tag{4.4}
$$

where $j=1,\dots,n$ and $m\in Z$. We used the representations

$$\psi^i=\lambda z_s+d, \qquad \psi'=\lambda' z'+d, \tag{4.5}$$

$$\lambda=c^{11}(\boldsymbol{x}_i)+ic^{12}(\boldsymbol{x}_i), \quad \lambda'=c^{11}(\boldsymbol{x}_j)+ic^{12}(\boldsymbol{x}_j), \quad d=d^1+id^2. \tag{4.6}$$

The form of the basis elements is determined due to the decomposition of the infinitesimals into the Laurent series. The commutation relations read

$$[k_j^m,k_j^l]=(m-l)k_j^{m+l}, \quad [\bar{k}_j^m,\bar{k}_j^l]=(m-l)\bar{k}_j^{m+l}, \quad [k_j^m,\bar{k}_j^l]=0. \tag{4.7}$$

Consider the vector space generated by

$$t_j^m=k_j^m\oplus\bar{k}_j^m \tag{4.8}$$

over C. With the commutation relations (4.7), we obtain the Witt algebra (a Lie algebra) $\mathfrak{w}_j=\mathfrak{w}(k)_j\oplus\mathfrak{w}(\bar{k})_j$. Here $\mathfrak{w}(k)_j$ and $\mathfrak{w}(\bar{k})_j$ are the Witt algebras generated by the basis elements k_j^m and $\bar{k}_j^m$ due to the commutators (4.7).

4.2 $\mathfrak{sl}(2)$ extended representation of the Witt algebra

We introduce the following subalgebras:

$$\mathfrak{w}(k)_{j_>}=\langle\{k_j^m\}\rangle_{m\geq -1}, \quad \mathfrak{w}(\bar{k})_{j_<}=\langle\{\bar{k}_j^m\}\rangle_{m\leq 1}. \tag{4.9}$$

Notice that the Witt algebra $\mathfrak{w}(k)_j$ is Z-graded Lie algebra by determining $\deg k_j^m=m$ that is $\mathfrak{w}(k)_{j_m}=\langle k_j^m\rangle$. It is clear that $\mathfrak{w}(k)_{j_>}$ and $\mathfrak{w}(k)_{j_<}$ are Z-graded Lie subalgebras of $\mathfrak{w}(k)_j$. Moreover, $\mathfrak{w}(k)_{j_>}$, $\mathfrak{w}(k)_{j_<}$ and $\mathfrak{w}(k)_j$ are simple Lie algebras [22]. Therefore, their non trivial representations exists. Among the subalgebras of $\mathfrak{w}(k)_j$ is a subalgebra generated by $\{k_j^{-1},k_j^0,k_j^{+1}\}$ that is isomorphic to $\mathfrak{sl}(2)$. We recall that $\mathfrak{sl}(2)$ is a simple Lie algebra and Cartan basis for it consists of a basis $\{e,f,h\}$ satisfy the commutation relations

$$[e,f]=h, \quad [h,e]=2e, \quad [h,f]=-2f. \tag{4.10}$$

Hence, the following triangular decomposition into Lie subalgebras holds:

$$\mathfrak{sl}(2)=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+, \tag{4.11}$$

where

$$\mathfrak{n}^-=\langle f\rangle, \quad \mathfrak{h}=\langle h\rangle, \quad \mathfrak{n}^+=\langle e\rangle. \tag{4.12}$$

There is a natural embedding

$$i:\mathfrak{sl}(2) \;\mapsto\; \mathfrak{w}(k)_j \tag{4.13}$$

that sends f to k_j^{-1}, h to $-2k_j^0$, and e to $-k_j^{+1}$ and induces an isomorphism of $\mathfrak{sl}(2)$ with $\mathfrak{w}(k)_{j_>}\cap\mathfrak{w}(k)_{j_<}$. They are interchanged by the Chevalley involution defined by $\Theta(k_j^i)=(-1)^{i+1}k_j^{-i}$.

The study of $\mathfrak{sl}(2)$ representations is the cornerstone of the representation theory of finite-dimensional Lie algebras and those of Witt have also been extensively studied. With this, it is natural to ask how the map i relates the representation theories of Witt and $\mathfrak{sl}(2)$. We mention a result [22] of whether an arbitrary $\mathfrak{sl}(2)$-module V admits a compatible Witt–module $\mathfrak{w}(k)_{j_>}$ structure which characterizes when a representation $\sigma:\mathfrak{sl}(2)\mapsto\mathrm{End}(V)$ can be extended to a Lie algebra representation $\Phi_>:\mathfrak{w}(k)_{j_>}\mapsto\mathrm{End}(V)$ where $\Phi_>$ is the linear map sending k_j^{-1} to $\sigma(e)$, k_j^0 to $-1/2\sigma(h)$, k_j^{+1} to $-\sigma(e)$ and k_j^m to $(1/m!)ad(\sigma(e))^m(T)$ for $m\geq 0$. Then, $\Phi_>$ is a Lie algebra representation if and only if the compatibility conditions are fulfilled [22]

$$\mathrm{ad}\big(\sigma(f)\big)(T)=3\sigma(e),\quad \mathrm{ad}\big(\sigma(h)\big)(T)=4T, \tag{4.14}$$

and there exists N such that for all $l\geq N$ there holds

$$\left[T,\frac{1}{(2l-1)!}\mathrm{ad}\big(\sigma(e)\big)^{2k-1}(T)\right]=-\frac{(2l-1)!}{\mathrm{ad}}\big(\sigma(e)\big)^{2l+1}(T). \tag{4.15}$$

Here ad is the adjoint representation of the Lie algebra. Since the Chevalley involution Θ in the Witt exchanges $\mathfrak{w}(k)_{j_>}$ and $\mathfrak{w}(k)_{j_<}$, the result produces an analogous one for the Witt module $\mathfrak{w}(k)_{j_<}$ and the representation $\Phi_<$. We will use the representation of $\mathfrak{w}(k)_j$

$$\Phi=\{\Phi_<,\sigma,\Phi_>\}. \tag{4.16}$$

4.3 Holomorphic fiber space

Consider the $\mathfrak{sl}(2)$-module V and a C-vector subspace $V_d\subset V$ of dimensional $d+1$

$$V_d=\langle x^d,x^{d-1}y,\dots,xy^d\rangle. \tag{4.17}$$

V_d is the set of homogeneous polynomials of total degree d. For $d=1$, the vector space V_1 consists of the polynomials of deg$\,=1$ and $\dim(V_1)=2$. The structure of $\mathfrak{sl}(2)$-module V_d is defined by

$$(x^ay^b)e=y\frac{\partial}{\partial y}(x^ay^b)=ax^{a-1}y^{b+1}, \tag{4.18}$$

$$(x^ay^b)f=x\frac{\partial}{\partial y}(x^ay^b)=bx^{a+1}y^{b-1}, \tag{4.19}$$

$$(x^ay^b)h=(a-b)x^ay^b \tag{4.20}$$

for $a+b=d$ and $0\le a,b\le d$. Therefore, the space V_1 is isomorphic to C. Further, we can think about the manifold X locally as a domain of the projective line $P_1(C)$. The $\mathfrak{sl}(2)$-module V is an associated vector fiber space and $V_1\subset V$ can be described by a family of the complex lines L_i, namely all multiples of $\lambda z_i,\lambda\in C$. Futher, the fiber space over $z_{(j)}$ is given by the gage transformation $g_i(z_{(j)})=\lambda z_i$ (associated with the group G_j which transforms the variables z_i). Then the gage transformation $g_i(z_{(j)})=\lambda^{m_i}z_i^{m_i}$ defines the fiber $L_i^{m_i}=L_i\otimes\cdots\otimes L_i$ (m_i times) [2]. Assuming $L_i^{-1}=L_i^*$, where L_i^* is the dual space, we can introduce the negative m. An arbitrary holomorphic gage transformation $f_i(z_{(j)})$ can be written as $f_i(z_{(j)})=f_+z_i^{m_i}f_-$ where $f_+=\sum a_m z_i^{m_i}$ with $m\le 0$ and $f_-=\sum a_m z_i^{m_i}$ with $m\ge 0$ due to the Laurent series representation. Therefore, arbitrary holomorphic fiber over $z_{(j)}$ associated with the group G_j which transforms the variables z_i, $i=1,\dots n$ is isomorphic to $L_i^{m_i}$. Hence, with respect to $z_1,\dots,z_n$ the holomorphic fiber looks like as the direct sum

$$E_j=L_1^{m_1}\oplus\cdots\oplus L_n^{m_n} \tag{4.21}$$

for $m_i<\infty$.

Consider the variable z_{n+1}, the group G_j acts on z_{n+1} as the scaling and translation transformations. With respect to the variables $\omega_1,\dots,\omega_{n+1},f_n,f_{n+1}$ the action of G_j is presented by the scaling transformations with prefactors given by formulas (3.28), (3.31) and (3.32), respectively. Therefore, the representations of such transformations are given by diagonal matrixes. Hence, the fiber over $\boldsymbol{x}_{(j)}$ admits the associated vector space

$$E_{G_j}=E_j\oplus C^*\oplus R^{n+1}\oplus S^n\oplus S^{n+1}, \tag{4.22}$$

where S^n and S^{n+1} are the unit spheres of the Banach spaces $L^1(\mathcal{M}^n,\nu_1\otimes\cdots\otimes\nu_n)$ and $L^1(\mathcal{M}^{n+1},\nu\otimes\cdots\otimes\nu_{n+1})$, respectively, where ν_i is the standard Lebesgue measure equals $d\omega_{(i)}$. Instead of PDFs f_n and f_{n+1}, we can consider the corresponding probability measures $\mu_n=f_n d\omega_{(1)}\wedge\cdots\wedge d\omega_{(n)}$ and $\mu_{n+1}=f_{n+1}d\omega_{(1)}\wedge\cdots\wedge d\omega_{(n+1)}$, respectively. It is easy to verify by the direct calculations that the group G_j acts on μ_n and μ_{n+1} as an invariant transformation. That is, the probability measures μ_n and μ_{n+1} are invariants of the group G_j. C^* is the complex line with respect to the multiplicative group which is the holomorphically trivial. Notice that, the cohomology group $H^1(U,L_i^{m_i})$ classifies the fibers $L_i^{m_i}$ over $U\subset P_1(C)$ and $\dim H^1(U,L_i^{m_i})=m_i-1$, see [2].

4.4 Covariant derivative

We consider the flow space X, the coordinate domain $\boldsymbol{x}_j\in U$ in X, the group G_j and an 1-differential form defined on U that takes the values in the Lie algebra $\mathfrak{w}(k)_j$

$$\theta_U=\sum_m t_j^m\otimes\theta_{U,m},\quad \theta_{U,m}=\sum_i \theta_{i,m}\mathrm{d}z^i_{(j)},\quad z^1_{(j)}=z_{(j)},\quad z^1_{(j)}=\bar{z}_{(j)}. \tag{4.23}$$

We also consider the representation Φ of $\mathfrak{w}(k)_j$. With an each vector field K on X, we associate the operator

$$\nabla_{\mathbf{z}_{(j)}} = \sum_s e_s^U \left[K + \sum_l B^U(K)_{ls} \right], \quad B^U(K) = \sum_m \Phi(t_j^m)\theta_{U,m}(K), \quad \mathbf{z}_{(j)} = (z_{(j)}, \bar{z}_{(j)}), \tag{4.24}$$

where $e_l^U : U \mapsto \pi_V^{-1}(U)$ is the basis of local sections, $\pi_V : V \mapsto C \times C$, see for details [24]. With this,

$$\theta_{i,m} = \theta_{U,m}\left(\frac{\partial}{\partial z^i_{(j)}}\right) \tag{4.25}$$

is the Yang-Mills field [24].

The curvature of the connection θ_U is defined by the formula

$$\begin{aligned} &\Gamma_U = d\theta_U + \frac{1}{2}[\theta_U, \theta_U], \\ &[\theta_U, \theta_U] = \left[\sum_m t_j^m \otimes \phi_m, \sum_n t_j^n \otimes \psi_n \right] = \sum_{m,n} \left[t_j^m, t_j^n \right] \otimes (\phi_m \wedge \psi_n), \end{aligned} \tag{4.26}$$

where d is the exterior differentiation operator, ϕ_m, ψ_n are scalar forms. Γ_U can be written in the form

$$\Gamma_U = \sum_m t_j^m \otimes \sum_{s<l} \Gamma_{sl}^m \mathrm{d}z^s_{(j)} \wedge \mathrm{d}z^l_{(j)}, \tag{4.27}$$

where

$$\Gamma_{sl}^m = \frac{\partial \theta_{l,m}}{\partial z^s_{(j)}} - \frac{\partial \theta_{s,m}}{\partial z^l_{(j)}} + \sum_{m,n} (m-n)(\theta_{s,m}\theta_{l,n} - \theta_{l,m}\theta_{s,n}). \tag{4.28}$$

Here $F_{sl}^m = -\Gamma_{sl}^m$ refers to the gauge field [24].

5 Statistical form of Brenier's variational functional

We are interested in solving f_n-equation (2.8) which is unclosed that leads so far hardly found any solutions. A first step in that direction is using the variational Brenier principle (VPB) [6] to close the f_n-equation without forcing $\mathcal{F}=0$ i.e. for the equilibrium state of turbulence. We refer to f_n as the Euler quantity f_n^E. Specifically, we present the generalized Brenier principle in the terms of optimal statistical ensemble. Brenier's concept – a representation of solutions to the equations of ideal incompressible fluids in terms of probability measures on the set of Lagrangian trajectories in the case of their stochasticity, is a generalization of Arnold's principle of least action (APLA) [1] of finding smooth solutions of Euler's equations. Brenier's formulation is essentially a probabilistic generalization of Arnold's principle. It consists in minimising a suitably defined average kinetic energy among a wide class of stochastic processes, the so-called generalised Lagrangian flows rather than among the pure deterministic class of smooth mappings. The minimising generalised flow is a probability distribution on the space of Lagrangian paths. The solution is obtained by prescribing boundary conditions in time, which amounts to specifying the Lagrangian transition probabilities from the initial to the final time [6].

5.1 Generalized variational functional

According to Arnold's principle of least action, the motion of an ideal fluid is realized as an extremal of the functional – the kinetic energy of fluid particles integrated over time $t\in[0,T]$ for $T\ll 1$. It is well known [29], that in order to reproduce the characteristic features of turbulent flows, Arnold's principle of least action must be weakened. One of the approaches is the VPB, which is formulated as finding the extremum of the Brenier functional

$$\mathcal{B}[\mu]=\int \mu[\mathcal{D}\gamma]\mathcal{S}[\gamma] \ \rightarrow \ \inf, \quad \mu|_{t=0}=\mu_0, \quad \mu|_{t=T}=\mu_T, \tag{5.1}$$

where $\mu[\mathcal{D}\gamma]$ – uniformal in t the probability measure (generalized flows) on Lagrangian random trajectories $t\mapsto\gamma(\mathbf{a},t)$ of the flow $(\mathbf{a},t)\mapsto\gamma(\mathbf{a},t)$ under the initial conditions $\gamma(\mathbf{a},0)=\mathbf{a}$ and boundary conditions

$$\mu[\mathcal{D}\gamma](t=0)=\mu_0, \quad \mu[D\gamma](t=T)=\mu_T, \tag{5.2}$$

which represent the probability distributions of a trajectory passing through given points at given moments in time $t=0, t=T, \mathcal{D}\gamma$ is a vector dual to the tangent vector γ_t. The variational functional is defined as

$$\mathcal{S}[\gamma]=\frac{1}{2}\int_0^T\int_D|\gamma_t(\mathbf{a},t)|^2\mathrm{d}\mathbf{a}dt, \tag{5.3}$$

where $|\gamma_t(a,t)|^2$ is the square of the Riemannian length of tangent vectors, $\mathcal{S}[\gamma]$ is a random quantities. $\mu[\mathcal{D}\gamma]$ is the probability for a trajectory $\gamma_t(a,t)$ to pass through given points when t changes from 0 to T. The following properties of the problem (5.1), (5.2) are proved [6]:

a) the solution of the problem exists in the sense of a generalized flow;
b) the generalized variational principle realizes (classical) solutions of the Euler equation under the condition that $T\ll 1$, while $\mu_T=\delta(x-X(a,T)da$, where $(a,t)\mapsto X(a,t)$ is the Lagrangian flow of motion of an ideal fluid and $X(a,T)$ is the position at time $t=T$, the measure $\mu[\mathcal{D}X]$ is singular on Lagrangian trajectories;
c) for $t>T$, the stochastic motions of an ideal fluid are determined by the probability measure (generalized flow) $\mu[\mathcal{D}\gamma]$.

VPB is a linear optimization problem in which both the objective function, i.e. $\mu[\mathcal{D}\gamma]$, and the constraints (5.2) depend linearly on the arguments that need to be minimized. Discrete flows for which analogues of the variational functional are defined can be presented by the permutation group, which was first introduced in [28] for approximating the Lagrangian flow and was considered as a model of the fundamental group of diffeomorphisms of an ideal incompressible fluid.

The characteristics of Eq. (2.8) describe the dynamics of the n Lagrangian particles $\boldsymbol{X}_{n(j)}(t)$

$$\frac{d}{dt}\boldsymbol{X}_{n(j)}(t)=\langle \boldsymbol{u}(\boldsymbol{x}_{(j)},t)|\omega_{(l)},\boldsymbol{x}_{(l)}\rangle\big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}},\qquad \boldsymbol{X}_{n(j)}(0)=\mathbf{a}_j, \tag{5.4}$$

$$\frac{d}{dt}\Omega_{n(j)}(t)=\langle \mathcal{F}(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle\big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}},\quad \Omega_{(l)}(0)=\omega^0_{(l)}, \tag{5.5}$$

where $j,l=1,\dots,n$. The subscript in the expression $\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}$ means that the statistics are calculated at the current position of the particle on the characteristic. The vorticity changes along the characteristic according to (5.5). In view of (5.5) for $\mathcal{F}=0$, the vorticity is transported along $\boldsymbol{X}_{n(l)}(t)$ without alternating the value. The characteristics $\boldsymbol{X}_{n(j)}$ ($Z_{n(j)}\in C$) in the complex variables $z_{(j)}$ are conformally invariantly transformed by the group G_j. The infinitesimal operator of the Lie algebra of the subgroup $H_j\subset G_j$ transforming $z_{(j)}$ and f^E_n is of the form

$$T_j=\psi^j(z_{(j)})\frac{\partial}{\partial z_{(j)}}+\bar{\psi}^j(\bar{z}_{(j)})\frac{\partial}{\partial \bar{z}_{(j)}}-\psi^j_{z_{(j)}}(z_{(j)})f^E_n\frac{\partial}{\partial f^E_n}-\bar{\psi}^j_{\bar{z}_{(j)}}(\bar{z}_{(j)})f^E_n\frac{\partial}{\partial f^E_n}. \tag{5.6}$$

Eq. (5.4) describes the dynamics of the j-th Lagrangian particle on the j-th component of the n-dimensional complex space $C^n=C_{(1)}\times\cdots\times C_{(n)}$, where $C_{(j)}\simeq C$. Thus, $Z_{(n)j}(t)=X^1_{n(j),t}(t)+iX^2_{n(j),t}(t)$ is a curve on $C_{(j)}$ along which $\omega^0_{(j)}$ is preserved. The length element (metric) in $C_{(j)}$ is defined by the function Λ

$$dl^2_{(j)}=\Lambda^2(z_{(j)},\bar{z}_{(j)})dz_{(j)}d\bar{z}_{(j)}. \tag{5.7}$$

The infinitesimal operator of the widest group of transformations that invariantly transforms $dl^2_{(j)}$, hence the length of the characteristic $Z_{n(j)}(t)$, coincides with $T_{(j)}$ [15]. Thus, the condition $\Lambda^2=f^E_n$ defines the Riemannian length of the velocity vector $dZ_{n(j)}(s)/dt$ that is invariant with respect to the action of the group H_j.

Let us formulate a statistical analogue of the VPB. The kinetic energy of the Lagrangian flow $\boldsymbol{X}_{n(j)}(\mathbf{a}_j,t)$, integrated over t, is equal to

$$\mathcal{S}[\boldsymbol{X}_{n(j)}]=\frac{1}{2}\int_0^T\int_D\left|\frac{d}{dt}\boldsymbol{X}_{n(j)}(\mathbf{a}_j,t)\right|^2 \mathrm{d}\mathbf{a}_j dt. \tag{5.8}$$

Taking into account (2.9), (2.10) and the conformal form of the metric $dl^2_{(j)}$, i.e. (5.7), we obtain

$$\begin{aligned}\mathcal{S}[\boldsymbol{X}_{n(j)}]&=\int_0^T\int_{D_j}\left|\frac{d}{dt}\boldsymbol{X}_{n(j)}(t)(\mathbf{a}_j,t)\right|^2\mathrm{d}\mathbf{a}_j dt \\ &=\int_0^T\int_{D_j} f^E_n\big(\langle u(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle^2+\langle v(\boldsymbol{x}_{(j)},t)|\{\omega_{(l)},\boldsymbol{x}_{(l)}\}\rangle^2\big)\big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}}\mathrm{d}\mathbf{a}_j dt\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
=\int_0^T\int_{D_j}\Bigg[& f_n^E\left(\int \mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^1(\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}^E}{f_n^E}\right)^2 \\
&+f_n^E\left(\int \mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^2(\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}^E}{f_n^E}\right)^2\Bigg]\Bigg|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}}\mathrm{d}\mathbf{a}_j dt,
\end{aligned}
$$

where $\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}$ specifies the Lagrangian representation of the velocity field and $\mathcal{S}[\boldsymbol{X}_{n(j)}]$ is a Lagrangian random variable. The Lagrangian PDF of the vortex field reads

$$
\begin{aligned}
& f_n^L(\boldsymbol{x}_{(1)},\mathbf{a}_1,\omega_{(1)},\ldots,\boldsymbol{x}_{(n)},\mathbf{a}_n,\omega_{(n)},t) \\
&=\left\langle\prod_{i=1}^n\delta\big(\boldsymbol{x}_{(i)}-\boldsymbol{X}_{n(i)}(\mathbf{a}_i,t)\big)\delta\big(\omega_{(i)}-\Omega_{(i)}(\mathbf{a}_i,t)\big)\right\rangle. \qquad (5.10)
\end{aligned}
$$

The Brenier functional of the statistical ensemble of the random variable $\mathcal{S}[\boldsymbol{X}_{n(j)}]$ is defined by the formula

$$
\int\mu[\mathcal{D}\boldsymbol{X}_n]\mathcal{S}[\boldsymbol{X}_{n(j)}]=\int\mathrm{d}\mathbf{a}_1\ldots\mathrm{d}\mathbf{a}_n f_n^L\mathcal{S}[\boldsymbol{X}_{n(j)}], \qquad (5.11)
$$

where $\mu[\mathcal{D}\boldsymbol{X}_n]$ is the double stochastic measure [6]

$$
\mu[\mathcal{D}\boldsymbol{X}_n]=\left\langle\prod_{j=1}^n\delta\big(\boldsymbol{x}_{(j)}-\boldsymbol{X}_{n(j)}(\mathbf{a}_j,t)\big)\delta\big(\omega_{(j)}-\Omega_{(j)}(\mathbf{a}_j,t)\big)\right\rangle\mathrm{d}\mathbf{a}_1\ldots\mathrm{d}\mathbf{a}_n. \qquad (5.12)
$$

Integrating (5.1) with respect to $\mathrm{d}\mathbf{a}_1\ldots\mathrm{d}\mathbf{a}_n$, and using the formula (see [11])

$$
\int\mathrm{d}\mathbf{a}_1\ldots\mathrm{d}\mathbf{a}_n f_n^L=f_n^E(\boldsymbol{x}_{(1)},\omega_{(1)},\ldots,\boldsymbol{x}_{(n)},\omega_{(n)},t)\equiv\left\langle\prod_j^n\delta\big(\omega_{(j)}-\omega(\boldsymbol{x}_{(j)},t)\big)\right\rangle, \qquad (5.13)
$$

we obtain

$$
\int\mu[\mathcal{D}\boldsymbol{X}_n]\mathcal{S}[\boldsymbol{X}_{n(j)}]=f_n^E\mathcal{S}[\boldsymbol{X}_{n(j)}]. \qquad (5.14)
$$

Taking into account that f_n^E is constant along the characteristic $\boldsymbol{X}_{n(j)}(t)$, i.e. equals $f_n^{E^0}$ which is given at the initial time, the right-hand side (5.14) reads

$$
f_n^E\mathcal{S}[\boldsymbol{X}_{n(j)}]=\frac{1}{2}\int_0^T f_n^{E^0}\int_{D_j}\left[f_n^{E^0}\left(\int A_n^1\right)^2+f_n^{E^0}\left(\int A_n^2\right)^2\right]\Bigg|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}}\mathrm{d}\mathbf{a}_j dt,
$$

where

$$
\begin{aligned}
A_n^1&=\mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^1(\boldsymbol{X}_{(j)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}^E}{f_n^{E^0}}, \\
A_n^2&=\mathrm{d}\boldsymbol{x}_{(n+1)}\mathrm{d}\omega_{(n+1)}\omega_{(n+1)}\alpha^2(\boldsymbol{X}_{(j)}-\boldsymbol{x}_{(n+1)})\frac{f_{n+1}^E}{f_n^{E^0}}.
\end{aligned}
$$

Thus, the Brenier functional has the form

$$\mathcal{B}[f^E_{n+1}]=\frac{1}{2}\int_0^T f_n^{E^0}\int_{D_j} f_n^{E^0}\left[\left(\int A_n^1\right)^2+\left(\int A_n^2\right)^2\right]\mathrm{d}\mathbf{a}_j dt\Big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\}}. \tag{5.15}$$

The Brenier principle of least action for the functional (5.15) is formulated as follows:

$$\mathcal{B}[f^E_{n+1}] \longrightarrow \inf \quad \text{with the boundary conditions,} \tag{5.16}$$

$$f^E_{n+1}\big|_{t=0}=f^E_{n+1,0}, \quad f^E_{n+1}\big|_{t=T}=f^E_{n+1,T}, \tag{5.17}$$

where $f^E_{n+1,0}, f^E_{n+1,T}$ are the vortex field PDFs given at times $t=0, t=T$. Specifically, we have

$$\begin{aligned}
&f^E_{n+1}\Big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\},t=0}=\delta\big(\omega_{(n+1)}-\omega(\boldsymbol{x}_{(n+1)},0)\big)g_n\big(\{\omega^0_{(l)},\mathbf{a}_l\}\big),\\
&f^E_{n+1}\Big|_{\{\omega_{(l)},\boldsymbol{x}_{(l)}\}=\{\Omega_{(l)}(t),\boldsymbol{X}_{n(l)}(t)\},t=T}=\delta\big(\omega_{(n+1)}-\omega(\boldsymbol{x}_{(n+1)},T)\big)g_n\big(\{\omega^T_{(l)},\boldsymbol{X}_{n(j),f}\}\big),
\end{aligned}$$

where $\delta(\omega_{(n+1)}-\omega^0(\boldsymbol{x}_{(n+1)}))$ and $\delta(\omega_{(n+1)}-\omega^T(\boldsymbol{x}_{(n+1)}))$ are the PDFs at times $t=0$ and $t=T$,

$$\begin{aligned}
g_n\big(\{\omega^0_{(l)},a_l\}\big)&=\prod_{j=1}^n\delta\big(\omega_{(j)}-\omega^0(a_j)\big),\\
g_n\big(\{\omega^T_{(l)},\boldsymbol{X}_{n(l),f}\}\big)&=\prod_{j=1}^n\delta\big(\omega_{(j)}-\omega^T(\boldsymbol{X}_{n(j),f})\big).
\end{aligned}$$

The condition of uniformity of the associated probability measure μ with respect to t is a condition on the characteristics $\boldsymbol{X}_{n(j)}(t)$. The equality $d\Omega_{n(j)}(t)/dt=0$ along $\boldsymbol{X}_{n(j)}(t)$ leads to independence of μ with respect to t on the characteristics. Therefore, the uniformity condition is satisfied. The statistical formulation of the VPB becomes the APLA as $n\to\infty$ ($j\to\infty$). Indeed, formally passing to the limit as $n\to\infty$ we have

$$\big(\boldsymbol{X}_{n(j)}(\mathbf{a}_j,t),\Omega_{n(j)}(\mathbf{a}_j,t)\big) \;\to\; \big(\boldsymbol{X}(\mathbf{a},t),\Omega(\mathbf{a},t)\big),$$

where $\mathbf{a}$ fills D_j. Eqs. (5.4), (5.5) converge to the vorticity equation in the Lagrangian formulation (see [12, Formulas (52a), (52b)])

$$\begin{aligned}
&\frac{d}{dt}\boldsymbol{X}(a,t)=\int d\mathbf{a}'\Omega(\mathbf{a}',t)\big(\alpha^1\big(\boldsymbol{X}(\mathbf{a},t)-\boldsymbol{X}(\mathbf{a}',t)\big)+\alpha^2\big(\boldsymbol{X}(\mathbf{a},t)-\boldsymbol{X}(\mathbf{a}',t)\big)\big),\\
&\frac{d}{dt}\Omega(\mathbf{a},t)=0 \quad \text{along} \quad \boldsymbol{X}(a,t).
\end{aligned}$$

In this case, the right side (5.15) converges to

$$\int_0^T dt\left[\left(\int_{D_j}\mathrm{d}\mathbf{a}'\Omega(\mathbf{a}',t)\alpha^1\big(\boldsymbol{X}(\mathbf{a},t)-\boldsymbol{X}(\mathbf{a}',t)\big)\right)^2+\left(\int_{D_j}\mathrm{d}\mathbf{a}'\Omega(\mathbf{a}',t)\alpha^2\big(\boldsymbol{X}(\mathbf{a},t)-\boldsymbol{X}(\mathbf{a}',t)\big)\right)^2\right]$$

with the Biot-Savart kernel. Notice that the integral above coincides with the kinetic energy of the fluid volume integrated over t. The verification of fulfillment of the corresponding boundary conditions is elementary. The resulting variational boundary value problem for the functional (5.3) with given configurations of the fluid at $t=0$ and $t=T$ [28] is not a classical one, in which the initial position of the trajectory of the fluid particle and its velocity are specified.

6 Conclusion and outlook

The main aim of the proposed research was to deliver a through modern symmetry analysis of statistics for the 2D optical turbulence in defocusing media several methods of a gauge theory in the statistical turbulence. Focusing on the universal nonlinear and statistical phenomena, which are common for these systems – optical turbulence, quantum fluids, turbulent cascades, vortices – and exploiting the fact that these systems in the quantum fluid approximation of the $2D$ defocusing NLSE share the core hydrodynamical turbulence, an infinite chain of the Lundgren-Monin-Novikov equations for the multi-point PDFs of the weighted vorticity field has been used.

In this research, we have done more for the symmetry transformations G calculated in [13], which form a Lie group of the f_n-equation of the LMN chain. Specifically, we completely described the corresponding Lie algebra and this is the Witt-type algebra. The representation of the Witt-type algebra constructed was given using the $\mathfrak{sl}(2)$-module V, which has been explored to describe the holomorphic fiber over the basis $z_{(j)}$. With this, the main steps for building a gauge theory for the LMN chain have been presented. In reality, the fiber space $\mathcal{P}$ introduced over the flow space X where the fiber is a Lie group G_j over the base $z_{(j)}$, the associated vector space is constructed in Section 4.3, the holomorphic fiber looks like as the direct sum (4.21), the gauge potential is defined by the formula (4.23) and finally, the curvature tensor is determined by (4.26). The conformal symmetry property of G enables us to completely describe the fiber over the base $z_{(j)}$ – the associated vector space presented by formula (4.21). We need these steps to recast the LMN chain in the terms of connectedness of the fiber space $\mathcal{P}$ with studying the geometry of $\mathcal{P}$ in a separate analysis.

The results obtained in our studies will be important for applications of the conformal transformations for manipulating light, which are especially attractive due to the work by C. Wan *et al.* [35]. They reported the experimental observation of photonic toroidal vortex generated by the use of conformal mapping. The visual preparation of such an intriguing state of light demonstrates a stochastic iso-intensity profile of the photonic toroidal vortex in different views. The resulting wave field has a helical phase that twists around a closed loop. A stochastic behavior appeals to helical vortex structures which arise as unstable modes in swirling jets. Helical flows present significant interest because all quantities depend on the two spatial variables and therefore dimensionally reduce the underlying equations of motion. However, since there are still three independent velocity components in addition to the two independent variables, the corresponding turbulence is positioned between 2D and 3D and is often referred to as 2.5D turbulence. We believe that the methods which we presented here for 2D turbulence are also valid for 2.5D case. The development of a universal theory for turbulent flows is still an open research topic despite the high practical relevance of turbulent flows. However, the statistical character of turbulent flows is undisputed.

One more application of the group G was using the variational Brenier principle to close the f_n-equation. To calculate the functional defined (5.1), we need a Riemannian metric to determine (5.8) for the Lagrangian flow $X_{n(j)}$. This metric should be invariant under the invariant transformations of $X_{n(j)}$. The general form of Riemannian metrics on the complex line $C_{(j)}$ is defined by formula (5.7). The infinitesimal operator (5.6) of the widest group of transformations that invariant transforms (5.7), which is conformal with respect to the spatial variables, coincides with the infinitesimal operator generated by subgroup H_j of the group G_j. With this, $\Lambda^2 = f_n^E$ defines the Riemannian length of the velocity vector. Moreover, it enables us to equip a set of fluid particles defined by the configuration of the n fluid particles that are moving within the conditionally averaged velocity field by a Riemannian metric. It gives more insight into the cooperative behavior of the fluid particles and the geometry of their configurations or the shape evolution of the Lagrangian cloud.

Acknowledgments

This work was supported by the National Key R&D Program of China (Grant No. N2020YFE0204200) and by the Sino-Russian Mathematical Center.

The author would like to thank the Sino-Russian Mathematical Center PKU for the hospitality.

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