

Optical Wave Turbulence: The Conformal Symmetry Transformations of Statistics of the Quantum Fluids of Light

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Abstract. We use the quantum fluid analogy [Nazarenko *et al.*, Phys. Rev. E, 92, 2015] for coherent structures (vortices or solitons) interacting among themselves and with the random wave component. This is performed for 2D defocusing media based on quantum fluid approximation of the two-dimensional nonlinear Schrödinger equation in the statistical frame. With this, the Lundgren-Monin-Novikov infinite chain of equations for the n -point density function f_n for the vorticity field is used. The conformal group of symmetry transformations calculated [Grebenev *et al.*, Theor. Math. Phys., 217(2), 2023] is applied to implement several elements of a gauge theory in the conformal transformation optics. Finally, we demonstrate how to use the variational generalized Brenier principle [Brenier, J. Am. Math. Soc., 2, 1989] together with the conformal invariance of statistics to close the infinite chain of Landgren-Monin-Novikov equations.

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Key words: Optical wave turbulence, n -point statistics of a vortex field, gauge transformations, conformal invariance, generalized Brenier variational functional.

1 Introduction

We study an exciting emerging area: turbulence in nonlinear optical systems [8, 30–32]. It deals with such universal features of turbulence as the invariance of statistics of vorticity field shared by a great variety of applications, ranging from quantum to classical. The distinct property typical for quantum fluids of light is a tangle of randomly moving quantized vortex lines. Nonlinear optics is a hugely broad subject, covering many optical applications and a variety of optical media. For instance, it concerns with nonlinear optical fibres, liquid crystals, photo-refractive crystals, and light propagation through atomic

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vapours [25, 26]. Moreover, there are close analogies of optical processes with hydrodynamic turbulence in classical and condensed matter fluids such as superfluids [5]. For example, in both optics and condensed matter fluids, there exist vortex-like structures, shock waves, and weakly interacting random waves whose dynamics and statistics have similarities to random waves that appear on the water surface [5]. Optical systems, where a laser-produced light propagates in a non-linear medium, have one distinct feature in common – the coherence of electromagnetic waves [20]. Coherence is commonly considered the opposite to randomness or turbulence, which is why, perhaps, chaotic behavior was traditionally considered undesirable in such experiments until recently. The situation started to change drastically when it was realized that non-linear light exhibits interesting fluid-like behaviors with vortex structures and waves involved in complicated random motions having all essential features of classical turbulence [7].

At most fundamental perspective, the turbulence theory aims to understand the complex random interactions between turbulent fluctuations in systems containing a vast number of degrees of freedom. Turbulence is ubiquitous in nature, it is observed in both the classical hydrodynamics, astrophysics and magneto-hydrodynamics, superfluids and Bose-Einstein condensate (BEC), weakly interacting waves, and nonlinear optics. Turbulence is one of the most important unsolved problems in physics, and developing a realistic description of the dynamics and statistics of its constituent entities, such as waves, vortices and other type of coherent structures, the transitions to turbulence, and the influence of turbulence on the other processes in natural and technological conditions would enable us to understand and control this important phenomenon which would bring about significant benefits to industry. It is natural to study nonlinear optics systems together because, at the most basic level, it share a nonlinear model based on the nonlinear Schrödinger equation (NSLE). Often the NSLE model is modified by adding a trapping potential term, terms accounting for dissipation, nonlocality of interaction, finite relaxation time, etc.

Another strong motivation for studying turbulent optical systems and designing respective experiments arises from strong links with a new area of non-equilibrium statistical mechanics – the distribution of n -point probability density functions (PDFs) f_n for the vorticity (scalar) fields, which is defined as a system of random vortexes with a broad frequency spectrum, which are involved in statistical nonlinear interactions. This is described by the Navier-Stokes equations in the statistical frame, i.e. the Lundgren-Monin-Novikov hierarchy [12] for f_n using the hydrodynamic analogy for NSLE. Simulations of 2D defocusing optical turbulence have been performed in various settings using the 2D defocusing NSLE model, i.e. with and without initial condensate present, forcing at large or at small scales. Simulations without an initial condensate and forcing at small scales exhibited presence of an inverse cascade in agreement with wave turbulence theory [21, Chapter 8.2, Remarks 8.2.2 and 8.2.3] predictions. Such a strongly nonequilibrium vortex condensation eventually lead to a strongly turbulent stage with interacting and annihilating vortices. It was noted that presence of an acoustic component facilitates the vortex annihilations, in agreement with a scenario which says that rapid cooling