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Superconvergence Points of Several Polynomial and Nonpolynomial Hermite Spectral Interpolations

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Abstract. In this paper, we analyze the superconvergence properties for spectral interpolations by Hermite polynomials and mapped Hermite functions. At the superconvergence points, the $(N-k)$-th term in the Hermite spectral interpolation remainder for the $(k+1)$-th derivatives vanish. To solve multi-point weakly singular nonlocal problems, we previously introduced mapped Hermite functions (MHFs), which are constructed by applying a mapping to the Hermite polynomials. We prove that the superconvergence points of the spectral interpolations based on MHFs for the $(k+1)$-th derivatives are the zero points of the $(N-k)$-th term. Additionally, due to the rapid growth of the logarithmic function at the endpoints 0 and 1, we further propose generalized mapped Hermite functions (GMHFs). We develop basic approximation theory for these new orthogonal functions and prove the projection error and interpolation error in the L^2-weighted space using the pseudo-derivative. We demonstrate that the superconvergence points of the spectral interpolations based on both MHFs and GMHFs for the $(k+1)$-th derivative are the zero points of the $(N-k)$-th term. Numerical experiments confirm our theoretical results.

AMS subject classifications: 65N35, 65M70, 41A05

Key words: Superconvergence points, interpolation, Hermite polynomials, mapped Hermite functions, generalized mapped Hermite functions.

1 Introduction

The study of superconvergence phenomenon for h-version methods has had a significant impact on scientific computing, particularly on a posteriori error estimates and adaptive methods. As for the p-version methods and spectral methods, the initial studies, presented in [34, 42, 43], discussed some special and simple cases. The superconvergence

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properties of some high-order orthogonal polynomial interpolations were studied, and the superconvergence points were identified in [44]. These results were subsequently extended to general Jacobi-Gauss-type interpolation in [35]. Zhao and Zhang [45] investigated the superconvergence points of spectral interpolation involving fractional derivatives. The superconvergence properties of the Riemann-Liouville fractional derivative of Hermite interpolations were explored in [10]. Xiang *et al.* [38] rigorously showed that the Jacobi expansion for a more general class of ϕ-functions also exhibits a local supconvergence behaviour.

The Hermite spectral interpolation, especially in the spectral Galerkin method [18] and numerical quadratures, is widely applied in mathematical models involving nonlocal operators such as fractional integrals, fractional Laplacians, and nonlocal Laplacians. These methods have proven to be of great value and superior to conventional models in simulating many abnormal physical phenomena and engineering processes [9, 11]. We remark that there has been much interest in Hermite spectral methods for PDEs involving standard or integral fractional Laplacian in unbounded domains [21, 30]. In these methods using Hermite polynomial/function-based approaches, selecting more accurate points is a critical concern. One of the efforts in this work is dedicated to identifying superconvergence points for the $(k+1)$-th derivatives of the interpolants using Hermite polynomials/functions.

Nonlocal models, such as weakly singular integral equations (WSIEs), have been demonstrated to be more effective in modeling complex systems in physics, finance, and other fields [1, 17, 23, 24]. Due to the nonlocal nature of the weakly singular integral operator, existing methods such as collocation methods, Petrov-Galerkin methods, discontinuous Galerkin methods, and fast algorithms [2–5, 12, 20, 29, 31, 33, 36, 39, 40] have been applied. Spectral methods are promising candidates for solving WSIEs, as their global nature aligns well with the nonlocal definition of integral operators. Using integer-order orthogonal polynomials as basis functions, spectral methods [8, 16, 19, 25, 27] help reduce the memory cost associated with the discretization of fractional derivatives. Furthermore, to address singularities, the authors in [6, 7, 15, 41] have developed suitable basis functions.

One major challenge is to construct an effective basis for a spectral scheme to handle singularities. With a suitable basis, one can then analyze the approximation error to identify superconvergence points. The mapped Hermite spectral interpolation [37], which is based on mapped Hermite functions constructed by applying a mapping to Hermite polynomials, is specifically designed to match the multiple singularities present in the underlying solutions of weakly singular problems. In [37], we proposed a MHFs-spectral collocation method and a MHFs-smoothing transformation method to solve the two-point weakly singular Fredholm-Hammerstein integral equation directly, rather than indirectly by splitting the Fredholm integral kernel into two Volterra integral kernels. However, the mapping $\alpha\log(x/(1-x))$ used in the mapped Hermite functions grows rapidly near the endpoints $x=0$ and $x=1$, which may affect the accuracy in some situations. To address this issue, in this work we propose the generalized mapped Hermite functions