CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0035

Vol. **x**, No. x, pp. 1-38
xx 2025

Multistability of Bi-Reaction Networks

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Received 4 August 2024; Accepted 17 July 2025

Abstract. We provide a sufficient and necessary condition in terms of the stoichiometric coefficients for a bi-reaction network to admit multistability. Also, this result completely characterizes the bi-reaction networks according to if they admit multistability.

AMS subject classifications: 92C40, 92C45

Key words: Reaction networks, mass-action kinetics, multistationarity, multistability.

1 Introduction

This work addresses the multistability problem for the dynamical systems arising from the bio-chemical reaction networks (under mass-action kinetics). The problem is how to efficiently determine if a reaction network admits at least two stable positive steady states in the same stoichiometric compatibility class. Multistability is important in mathematical biology since it widely exists in the decision-making process and switch-like behavior in cellular signaling (e.g. [1, 9, 13, 19, 30]). In practice, one way to detect multistability is to first find nondegenerate multistationarity (i.e. to check if the network admits more than one positive nondegenerate steady state). Usually, one can obtain two stable steady states if the number of positive nondegenerate steady states is at least three (e.g. [14, 20, 28]). Generally, deciding multistationarity/multistability or computing the witnesses (i.e. a choice of parameters for which the network exhibits multistationarity/multistability) is challenging because the problem is known to be a special real quantifier elimination problem (that means we want to efficiently obtain the information of

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real solutions of a semi-algebraic system, e.g. [5,15]). However, there indeed exists a collection of efficient/practical methods for detecting multistationarity (e.g. [7,16,18,24]). Most of these approaches are to check if the determinant of a certain Jacobian matrix changes sign [2,6,8,10,23,29].

One big goal in the area of reaction network is to look for the "explicit" criteria. That means we hope to tell the dynamical behaviors of a network by reading the network itself without doing any expensive computations. One typical result, which makes the big goal realistic, is the well-known deficiency zero theorem and the deficiency one theorem [11]. So far, such explicit criteria for detecting multistationarity/multistability are only known for small networks with one species or up to two reactions (possibly reversible) [17,25]. For instance, in [17], the authors completely characterized one-species networks by "arrow diagrams", and the number of (stable) steady states can be read off by looking at the existence of T-alternating subnetworks with certain type of arrow diagrams. Later, in [21], the criterion for multistationarity described by arrow diagrams is extended to more general networks with one-dimensional stoichiometric subspaces. Since for the one-dimensional networks, admitting at least three positive steady states is a necessary condition for admitting multistability (e.g. [27, Theorem 3.4]), the explicit criterion for admitting three positive steady states (described by "bi-arrow diagrams") is studied in [26]. Also, the authors of [26] has completely characterized the stoichiometric coefficients of the bi-reaction networks that admit at least three positive steady states. We remark that in the point of view of real algebraic geometry, an explicit criterion for multistationarity/multistability is essentially an explicit criterion for deciding number of real solutions of a special class of semi-algebraic systems. Some related recent work is the extension of the Descartes' rule of signs for the high dimensional algebraic systems (e.g. [4,12]), which can also be applied to the steady-state systems arising from bio-chemical reaction networks.

In this paper, we focus on the bi-reaction networks that admit finitely many positive steady states. The main result is an explicit criterion for deciding multistability of bi-reaction networks (see Theorem 3.1). By this result, we completely classify all nontrivial bi-reaction networks according to if they admit multistability or not (here, a "nontrivial" network means this network admits at least one positive steady state). This work can be viewed as an extension of [26], since in [26], all bi-reaction networks are classified according to if they admit at least three positive steady states (recall that admitting three positive steady states is a necessary condition for multistability). In general, it is a challenging problem to drive explicit criteria for determining multistability. Since we know the multistability can be lifted from a small sub-network to a large biochemical network [3], explicit criteria for small networks such as bi-reaction networks are important for making a breakthrough.

The rest of this paper is organized as follows. In Section 2, we recall the basic definitions and notions for the reaction networks and the multistationarity/multistability. In Section 3, we present the main theorem (a sufficient and necessary condition in terms of the stoichiometric coefficients for a bi-reaction network to admit multistability), and we illustrate how to use the theorem for deciding multistability by several examples. In Section 4, we present the proof of the main theorem by discussing several cases. In the supplementary materials†, we present a list of useful lemmas and their proofs, and we provide Maple files for the computations presented in Section 3.

2 Background

2.1 Chemical reaction networks

In this paper, we follow the standard notions on reaction networks used in [26,27]. A reaction network G (or network for short) consists of finitely many reactions

$$\alpha_{1j}X_1+\cdots+\alpha_{sj}X_s \xrightarrow{\kappa_j} \beta_{1j}X_1+\cdots+\beta_{sj}X_s, \quad j=1,2,\ldots,m, \tag{2.1}$$

where $X_1,\ldots,X_s$ denote s species, the stoichiometric coefficients α_{ij} and β_{ij} are non-negative integers, each $\kappa_j\in\mathbb{R}_{>0}$ is a rate constant corresponding to the j-th reaction, and we assume that

$$\forall j\in\{1,\ldots,m\}, \quad (\alpha_{1j},\cdots,\alpha_{sj})\neq(\beta_{1j},\cdots,\beta_{sj}). \tag{2.2}$$

The stoichiometric matrix of G, denoted by $\mathcal{N}$, is the $s\times m$ matrix with (i,j)-entry equal to $\beta_{ij}-\alpha_{ij}$. The stoichiometric subspace, denoted by S, is the real vector space spanned by the column vectors of $\mathcal{N}$.

The concentrations of the species $X_1,X_2,\ldots,X_s$ are denoted by $x_1,x_2,\ldots,x_s$, respectively. Note that x_i can be considered as a function in the time variable t. Under the assumption of mass-action kinetics, we describe how these concentrations change in t by the following system of ordinary differential equations (ODEs):

$$\dot{x}=\big(f_1(\kappa;x),\cdots,f_s(\kappa;x)\big)^\top:=\mathcal{N}\cdot\begin{pmatrix}\kappa_1\prod\limits_{i=1}^{s}x_i^{\alpha_{i1}}\\ \kappa_2\prod\limits_{i=1}^{s}x_i^{\alpha_{i2}}\\ \vdots\\ \kappa_m\prod\limits_{i=1}^{s}x_i^{\alpha_{im}}\end{pmatrix}, \tag{2.3}$$

where x denotes the vector $(x_1,x_2,\ldots,x_s)$, and κ denotes the vector $(\kappa_1,\ldots,\kappa_m)$. Note that for every $i\in\{1,\ldots,s\}$, $f_i(\kappa;x)$ is a polynomial in $\mathbb{Q}[\kappa,x]$.

A conservation-law matrix of G, denoted by W, is any row-reduced $d\times s$ matrix (here, $d:=s-\text{rank}(\mathcal{N})$), whose rows form a basis of $S^\perp$. Note that $\text{rank}(W)=d$. Especially, if the stoichiometric subspace of G is one-dimensional, then $\text{rank}(\mathcal{N})=1$ and $\text{rank}(W)=s-1$.

†https://github.com/65536-1024/one-dim

Note that the system (2.3) satisfies $W\dot{x}=0$, and any trajectory $x(t)$ beginning at a nonnegative vector $x(0)=x^0\in\mathbb{R}^s_{>0}$ remains, for all positive time, in the following stoichiometric compatibility class with respect to the total-constant vector $c:=Wx^0\in\mathbb{R}^d$:

$$\mathcal{P}_c:=\left\{x\in\mathbb{R}^s_{\geq 0}: Wx=c\right\}. \tag{2.4}$$

2.2 Multistationarity and multistability

For a given rate-constant vector $\kappa\in\mathbb{R}^m_{>0}$, a steady state of (2.3) is a concentration vector $x^*\in\mathbb{R}^s_{\geq 0}$ such that $f_1(\kappa,x^*)=\cdots=f_s(\kappa,x^*)=0$, where $f_1,\ldots,f_s$ are on the right-hand side of the ODEs (2.3). If all coordinates of a steady state x^* are strictly positive (i.e. $x^*\in\mathbb{R}^s_{>0}$), then we call x^* a positive steady state. We say a steady state x^* is nondegenerate if $\mathrm{Im}(\mathrm{Jac}_f(x^*)|_S)=S$, where $\mathrm{Jac}_f(x^*)$ denotes the Jacobian matrix of f with respect to x, at x^*. A steady state x^* is exponentially stable (or simply stable) if it is nondegenerate, and all non-zero eigenvalues of $\mathrm{Jac}_f(x^*)$ have negative real parts. Note that if a steady state is exponentially stable, then it is locally asymptotically stable [22].

Suppose $N\in\mathbb{Z}_{\geq 0}$. We say a network admits N (nondegenerate) positive steady states if there exist a rate-constant vector $\kappa\in\mathbb{R}^m_{>0}$ and a total-constant vector $c\in\mathbb{R}^d$ such that it has N (nondegenerate) positive steady states in the stoichiometric compatibility class $\mathcal{P}_c$. Similarly, we say a network admits N stable positive steady states if there exist a rate-constant vector $\kappa\in\mathbb{R}^m_{>0}$ and a total-constant vector $c\in\mathbb{R}^d$ such that it has N stable positive steady states in $\mathcal{P}_c$.

The maximum number of positive steady states of a network G is

$$cap_{pos}(G):=\max\left\{N\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}: G \text{ admits } N \text{ positive steady states}\right\}.$$

Similarly, we define

$$cap_{stab}(G):=\max\left\{N\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}: G \text{ admits } N \text{ stable positive steady states}\right\}.$$

We say a network admits multistationarity if $cap_{pos}(G)\geq 2$. We say a network admits multistability if $cap_{stab}(G)\geq 2$.

3 Main result

In this section, we focus on the bi-reaction network G

$$\begin{aligned}
&\alpha_{11}X_1+\cdots+\alpha_{s1}X_s \xrightarrow{\kappa_1} \beta_{11}X_1+\cdots+\beta_{s1}X_s,\\
&\alpha_{12}X_1+\cdots+\alpha_{s2}X_s \xrightarrow{\kappa_2} \beta_{12}X_1+\cdots+\beta_{s2}X_s.
\end{aligned} \tag{3.1}$$

First, we define the following sets of indices, which give a partition of the set $\{1,\ldots,s\}$:

$$S_1:=\{i:\alpha_{i1}>\alpha_{i2},\beta_{i1}>\alpha_{i1},1\leqslant i\leqslant s\},$$

$$\begin{aligned}
&S_2 := \{i : \alpha_{i1} < \alpha_{i2}, \beta_{i1} < \alpha_{i1}, 1 \leqslant i \leqslant s\},\\
&S_3 := \{i : \alpha_{i1} > \alpha_{i2}, \beta_{i1} < \alpha_{i1}, 1 \leqslant i \leqslant s\},\\
&S_4 := \{i : \alpha_{i1} < \alpha_{i2}, \beta_{i1} > \alpha_{i1}, 1 \leqslant i \leqslant s\},\\
&S_5 := \{i : \alpha_{i1} = \alpha_{i2} \text{ or } \beta_{i1} = \alpha_{i1}, 1 \leqslant i \leqslant s\}.
\end{aligned} \tag{3.2}$$

For each $i \in \{1,\ldots,s\}$, we define the following notions:

$$a_i := |\alpha_{i1} - \alpha_{i2}|, \quad \gamma_i := |\beta_{i1} - \alpha_{i1}|. \tag{3.3}$$

Theorem 3.1. *Given a bi-reaction network G (3.1) with a one-dimensional stoichiometric subspace, suppose $0 < cap_{pos}(G) < +\infty$.*

(a) If all the four sets S_1, S_2, S_3, S_4 are non-empty, then G admits multistability if and only if

$$\sum_{i \in S_1} a_i > \min_{i \in S_4}\{a_i\}, \quad \text{or} \quad \sum_{i \in S_2} a_i > \min_{i \in S_3}\{a_i\}. \tag{3.4}$$

(b) If there are exactly three of the four sets S_1, S_2, S_3, S_4 are non-empty, then G admits multistability if and only if one of the following four statements (1)-(4) holds:

(1) S_1, S_3 and S_4 are non-empty, and

$$\sum_{i \in S_1} a_i > \min_{i \in S_4}\{a_i\}. \tag{3.5}$$

(2) S_2, S_3 and S_4 are non-empty, and

$$\sum_{i \in S_2} a_i > \min_{i \in S_3}\{a_i\}. \tag{3.6}$$

(3) S_1, S_2 and S_3 are non-empty, and there exists a subset S_2^* of S_2 such that

$$\sum_{i \in S_3} a_i > \sum_{i \in S_2^*} a_i > \min_{i \in S_3}\{a_i\}. \tag{3.7}$$

(4) S_1, S_2 and S_4 are non-empty, and there exists a subset S_1^* of S_1 such that

$$\sum_{i \in S_4} a_i > \sum_{i \in S_1^*} a_i > \min_{i \in S_4}\{a_i\}. \tag{3.8}$$

(c) If there are exactly two of the four sets S_1, S_2, S_3, S_4 are non-empty, then G admits multistability if and only if one of the following two statements (1)-(2) holds:

(1) S_2 and S_3 are non-empty and there exists a subset S_2^* of S_2 such that

$$\sum_{i \in S_3} a_i > \sum_{i \in S_2^*} a_i > \min_{i \in S_3}\{a_i\}. \tag{3.9}$$

(2) S_1 and S_4 are non-empty, and there exists a subset S_1^* of S_1 such that

$$\sum_{i\in S_4} a_i > \sum_{i\in S_1^*} a_i > \min_{i\in S_4}\{a_i\}. \tag{3.10}$$

(d) If only one of the four sets S_1,S_2,S_3,S_4 is non-empty, then G admits no multistability.

Remark 3.1. Part (c) of Theorem 3.1 implies that for the following several cases: S_1 and S_2 non-empty, S_1 and S_3 non-empty, S_2 and S_4 non-empty, and S_3 and S_4 non-empty, there is no multistability. In fact, by [26, Theorem 6.1(b)], we know that for these cases, a one-dimensional bi-reaction network admits no more than 2 positive steady states. By [27, Theorem 3.4], admitting 3 positive steady states is a necessary condition for a one-dimensional network to exhibit multistability.

Example 3.1. The following examples illustrate how Theorem 3.1 works. In the supplementary materials‡, we also provide Maple files for the readers to verify the presented computations.

(a) Consider the following network:

$$G:\quad \begin{array}{rcl} 4X_1+X_2+X_3 & \rightarrow & 5X_1+X_4, \\ X_1+2X_2+X_4 & \rightarrow & 3X_2+X_3. \end{array}$$

It is straightforward to check that

$$\begin{array}{lllll} a_1=3, & a_2=1, & a_3=1, & a_4=1, & \\ S_1=\{1\}, & S_2=\{2\}, & S_3=\{3\}, & S_4=\{4\}, & S_5=\emptyset, \end{array}$$
$$\sum_{i\in S_1} a_i = a_1 = 3 > 1 = a_4 = \min_{i\in S_4}\{a_i\}.$$

So by Theorem 3.1(a), we have $cap_{stab}(G)\geq 2$. The steady-state system augmented with the conservation laws is

$$\begin{aligned} \kappa_1 x_1^4 x_2 x_3 - \kappa_2 x_1 x_2^2 x_4 &= 0, \\ -x_1 - x_2 - c_1 &= 0, \\ -x_1 - x_3 - c_2 &= 0, \\ x_1 - x_4 - c_3 &= 0. \end{aligned}$$

One can check that for $c_1=-2, c_2=-17/10, c_3=3/10, \kappa_1=1$, and $\kappa_2=1$, the network has 3 positive steady states

$$\begin{aligned} x^{(1)} &= (0.3293, 1.671, 1.371, 0.02930), \\ x^{(2)} &= (1.000, 1.000, 0.7000, 0.7000), \\ x^{(3)} &= (1.548, 0.4521, 0.1521, 1.248), \end{aligned}$$

where $x^{(1)}$ and $x^{(3)}$ are stable.

‡https://github.com/65536-1024/one-dim

(b)(i) Consider the following network:

$$G: \begin{array}{rcl} 3X_1+3X_2+2X_3+X_4 & \to & 4X_1+4X_2+X_3+2X_4, \\ X_1+X_2+X_3+4X_4 & \to & 2X_3+3X_4. \end{array}$$

It is straightforward to check that

$$\begin{array}{lllll} a_1=2, & a_2=2, & a_3=1, & a_4=3, & \\ S_1=\{1,2\}, & S_2=\emptyset, & S_3=\{3\}, & S_4=\{4\}, & S_5=\emptyset, \end{array}$$
$$\sum_{i\in S_1} a_i=a_1+a_2=4>3=a_4=\min_{i\in S_4}\{a_i\}.$$

So by Theorem 3.1(b)(1), we have $cap_{stab}(G)\geq 2$. The steady-state system augmented with the conservation laws is

$$\begin{aligned} &\kappa_1 x_1^3 x_2^3 x_3^2 x_4-\kappa_2 x_1 x_2 x_3 x_4^4=0, \\ &x_1-x_2-c_1=0, \\ &x_1+x_3-c_2=0, \\ &x_1-x_4-c_3=0. \end{aligned}$$

One can check that for $c_1=9/100, c_2=3, c_3=1/10, \kappa_1=1$, and $\kappa_2=2$, the network has 3 positive steady states

$$\begin{aligned} &x^{(1)}=(0.1448,0.05478,2.855,0.04478), \\ &x^{(2)}=(0.7442,0.6542,2.256,0.6442), \\ &x^{(3)}=(2.103,2.013,0.8967,2.003), \end{aligned}$$

where $x^{(1)}$ and $x^{(3)}$ are stable.

(b)(ii) Consider the following network:

$$G: \begin{array}{rcl} 2X_1+2X_2+X_3+X_4+X_5+3X_6 & \to & 3X_1+X_2, \\ X_1+3X_2+3X_3+2X_4 & \to & 4X_2+4X_3+3X_4+X_5+3X_6. \end{array}$$

It is straightforward to check that

$$\begin{array}{llllll} a_1=1, & a_2=1, & a_3=2, & a_4=1, & a_5=1, & a_6=3. \\ S_1=\{1\}, & S_2=\{2,3,4\}, & S_3=\{5,6\}, & S_4=\emptyset, & S_5=\emptyset. & \end{array}$$

Choose $S_2^*=\{2,3\}$. Note that

$$\sum_{i\in S_3} a_i=4>\sum_{i\in S_2^*} a_i=3>\min_{i\in S_3}\{a_i\}=1.$$

So by Theorem 3.1(b)(3), we have $cap_{stab}(G) \geq 2$. The steady-state system augmented with the conservation laws is

$$\begin{aligned}
&\kappa_1 x_1^2 x_2^2 x_3 x_4 x_5 x_6^3 - \kappa_2 x_1 x_2^3 x_3^3 x_4^2 = 0,\\
&x_1 + x_2 - c_1 = 0,\\
&x_1 + x_3 - c_2 = 0,\\
&x_1 + x_4 - c_3 = 0,\\
&x_1 + x_5 - c_4 = 0,\\
&3x_1 + x_6 - c_5 = 0.
\end{aligned}$$

One can check that for $c_1 = 101, c_2 = 101, c_3 = 1000, c_4 = 100, c_5 = 315, \kappa_1 = 1$, and $\kappa_2 = 72$, the network has 4 positive steady states

$$\begin{aligned}
x^{(1)} &= (32.09, 68.91, 68.91, 967.9, 67.91, 218.7),\\
x^{(2)} &= (86.24, 14.76, 14.76, 913.8, 13.76, 56.29),\\
x^{(3)} &= (97.55, 3.450, 3.450, 902.5, 2.450, 22.35),\\
x^{(4)} &= (99.54, 1.464, 1.464, 900.5, 0.4641, 16.39),
\end{aligned}$$

where $x^{(2)}$ and $x^{(4)}$ are stable.

(c) Consider the following network:

$$G: \quad \begin{array}{lcl} 3X_1 + 4X_2 + 5X_3 + 2X_4 & \rightarrow & 4X_1 + 5X_2 + 7X_3 + 3X_4 + X_5, \\ 2X_1 + 2X_2 + 4X_3 + 3X_4 + 2X_5 & \rightarrow & X_4. \end{array}$$

It is straightforward to check that

$$\begin{array}{lllll} a_1 = 1, & a_2 = 2, & a_3 = 1, & a_4 = 1, & a_5 = 2. \\ S_1 = \{1,2,3\}, & S_2 = \emptyset, & S_3 = \emptyset, & S_4 = \{4,5\}, & S_5 = \emptyset. \end{array}$$

Choose $S_1^* = \{2\}$. Note that

$$\sum_{i \in S_4} a_i = 3 > \sum_{i \in S_1^*} a_i = 2 > \min_{i \in S_4} \{a_i\} = 1.$$

So by Theorem 3.1(c)(1), we have $cap_{stab}(G) \geq 2$. The steady-state system augmented with the conservation laws is

$$\begin{aligned}
&\kappa_1 x_1^3 x_2^4 x_3^5 x_4^2 - 2\kappa_2 x_1^2 x_2^2 x_3^4 x_4^3 x_5^2 = 0,\\
&x_1 - x_2 - c_1 = 0,\\
&2x_1 - x_3 - c_2 = 0,\\
&x_1 - x_4 - c_3 = 0,\\
&x_1 - x_5 - c_4 = 0.
\end{aligned}$$

One can check that for $c_1{=}100, c_2{=}1, c_3{=}101, c_4{=}90, \kappa_1{=}1$, and $\kappa_2{=}328$, the network has 4 positive steady states

$$\begin{aligned} x^{(1)} &= (101.6, 1.588, 202.2, 0.5879, 11.59), \\ x^{(2)} &= (108.1, 8.081, 215.2, 7.081, 18.08), \\ x^{(3)} &= (128.2, 28.21, 255.4, 27.21, 38.21), \\ x^{(4)} &= (190.6, 90.62, 380.2, 89.62, 100.6), \end{aligned}$$

where $x^{(1)}$ and $x^{(3)}$ are stable.

4 Proofs

Assumption 4.1. Without loss of generality, for any network G defined in (3.1), we assume that the set S_5 of indices defined in (3.2) is empty throughout the rest of the paper. In fact, if $0 < cap_{pos}(G) < +\infty$, one can always construct a new network such that the new network G is dynamically equivalent to the original one and the set S_5 for the new network is empty (e.g. [27, Lemmas 5.1 and 5.2]).

We assume that any bi-reaction network G mentioned has the form (3.1). Notice that by Assumption 4.1, we have

$$\beta_{i1} - \alpha_{i1} \neq 0, \quad i = 1, \ldots, s. \tag{4.1}$$

If G has a one-dimensional stoichiometric subspace, then there exists $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$ such that

$$\begin{pmatrix} \beta_{12} - \alpha_{12} \\ \vdots \\ {}_{s2} - \alpha_{s2} \end{pmatrix} = \lambda \begin{pmatrix} \beta_{11} - \alpha_{11} \\ \vdots \\ \beta_{s1} - \alpha_{s1} \end{pmatrix}. \tag{4.2}$$

By [17, Lemma 4.1] (also, see [27, Lemma 4.2]), we assume that $\lambda < 0$ in (4.2) (otherwise, the network admits no positive steady state). By substituting (4.2) into $f_1, \ldots, f_s$ in (2.3), we have

$$f_i = (\beta_{i1} - \alpha_{i1}) \left(\kappa_1 \prod_{k=1}^{s} x_k^{\alpha_{k1}} + \lambda \kappa_2 \prod_{k=1}^{s} x_k^{\alpha_{k2}} \right), \quad i = 1, \ldots, s. \tag{4.3}$$

We define the steady-state system augmented with the conservation laws

$$h_1 := f_1 = (\beta_{11} - \alpha_{11}) \left(\kappa_1 \prod_{k=1}^{s} x_k^{\alpha_{k1}} + \lambda \kappa_2 \prod_{k=1}^{s} x_k^{\alpha_{k2}} \right), \tag{4.4}$$

$$h_i := (\beta_{i1} - \alpha_{i1}) x_1 - (\beta_{11} - \alpha_{11}) x_i - c_{i-1}, \quad i = 2, \ldots, s. \tag{4.5}$$

We solve x_i from $h_i=0$, and we get

$$x_i=\frac{(\beta_{i1}-\alpha_{i1})x_1-c_{i-1}}{\beta_{11}-\alpha_{11}},\quad i=2,\dots,s. \tag{4.6}$$

We introduce a new variable z and a new parameter μ_1 such that

$$x_1=(\beta_{11}-\alpha_{11})(z+\mu_1). \tag{4.7}$$

Then, the conservation laws (i.e. $h_i=0$) can be written as

$$x_i=(\beta_{i1}-\alpha_{i1})(z+\mu_i), \tag{4.8}$$

where

$$\mu_i:=\mu_1-\frac{c_{i-1}}{(\beta_{11}-\alpha_{11})(\beta_{i1}-\alpha_{i1})}.$$

If $h_1=0$, then by (4.4) we have

$$\prod_{k=1}^{s}x_k^{\alpha_{k1}-\alpha_{k2}}=-\frac{\lambda\kappa_2}{\kappa_1}.$$

So,

$$\sum_{k=1}^{s}(\alpha_{k1}-\alpha_{k2})\ln x_k=\ln\left(-\frac{\lambda\kappa_2}{\kappa_1}\right). \tag{4.9}$$

Notice that we can replace x_i with (4.8). So, we define the left-hand side of (4.9) as a new univariate function $g(z)$

$$g(z):=\sum_{i=1}^{s}(\alpha_{i1}-\alpha_{i2})\ln(\beta_{i1}-\alpha_{i1})(z+\mu_i). \tag{4.10}$$

For $i\in S_1,S_4$, we define $d_i:=\mu_i$. For $i\in S_2,S_3$, we define $d_i:=-\mu_i$. Recall that we have defined the notions a_i,γ_i in (3.3). So, we have

$$\begin{aligned}g(z)=&\sum_{i\in S_1}a_i\ln\gamma_i(z+d_i)-\sum_{i\in S_2}a_i\ln\big(-\gamma_i(z-d_i)\big)\\&+\sum_{i\in S_3}a_i\ln\big(-\gamma_i(z-d_i)\big)-\sum_{i\in S_4}a_i\ln\gamma_i(z+d_i).\end{aligned} \tag{4.11}$$

Notice that the domain of the function $g(z)$ is $I:=(\mathcal{L},\mathcal{R})$, where

$$\begin{aligned}&\mathcal{L}:=\begin{cases}\max\{-d_i\}_{i\in S_1\cup S_4}, & S_1\cup S_4\neq\emptyset,\\-\infty, & S_1=S_4=\emptyset,\end{cases}\\&\mathcal{R}:=\begin{cases}\min\{d_i\}_{i\in S_2\cup S_3}, & S_2\cup S_3\neq\emptyset,\\+\infty, & S_2=S_3=\emptyset.\end{cases}\end{aligned} \tag{4.12}$$

By (4.10) and (4.11), we have

$$\frac{dg}{dz}(z)=\sum_{i=1}^{s}\frac{\alpha_{i1}-\alpha_{i2}}{z+\mu_i}=\sum_{i\in S_1}\frac{a_i}{z+d_i}+\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}-\sum_{i\in S_4}\frac{a_i}{z+d_i}. \tag{4.13}$$

Lemma 4.1. *Given a network G (3.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G)<+\infty$. Let $g(z)$ and I be the function and the interval defined as in (4.11) and (4.12). Then, G admits multistability iff (if and only if) there exist $\{d_i\}_{i=1}^s\subset\mathbb{R}$ and $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $g'(z_1)<0$ and $g'(z_2)<0$, where these d_i's are distinct from each other.*

Lemma 4.2. *Given a network G (3.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G)<+\infty$. Let $g(z)$ and $I=(\mathcal{L},\mathcal{R})$ be defined as in (4.11) and (4.12).*

(i) For any given $\{d_i\}_{i=1}^s\subset\mathbb{R}$, if

$$\lim_{z\to\mathcal{L}^+} g(z)=+\infty,\quad \lim_{z\to\mathcal{R}^-} g(z)=-\infty,\quad \lim_{z\to\mathcal{L}^+}\frac{dg}{dz}(z)=-\infty,\quad \lim_{z\to\mathcal{R}^-}\frac{dg}{dz}(z)=-\infty,$$

then there exists $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $g'(z_1)<0$ and $g'(z_2)<0$ iff there exists $\widetilde{z}\in I$ such that $g'(\widetilde{z})>0$.

(ii) For any given $\{d_i\}_{i=1}^s\subset\mathbb{R}$, if

$$\lim_{z\to\mathcal{L}^+} g(z)=-\infty,\quad \lim_{z\to\mathcal{R}^-} g(z)=-\infty,\quad \lim_{z\to\mathcal{L}^+}\frac{dg}{dz}(z)=+\infty,\quad \lim_{z\to\mathcal{R}^-}\frac{dg}{dz}(z)=-\infty,$$

then there exists $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $g'(z_1)<0$ and $g'(z_2)<0$ iff there exist $\widetilde{z}_1,\widetilde{z}_2\in I$ ($\widetilde{z}_1<\widetilde{z}_2$) such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0,\quad \frac{dg}{dz}(\widetilde{z}_2)>0.$$

(iii) For any given $\{d_i\}_{i=1}^s\subset\mathbb{R}$, if

$$\lim_{z\to\mathcal{L}^+} g(z)=-\infty,\quad \lim_{z\to\mathcal{R}^-} g(z)=+\infty,\quad \lim_{z\to\mathcal{L}^+}\frac{dg}{dz}(z)=+\infty,\quad \lim_{z\to\mathcal{R}^-}\frac{dg}{dz}(z)=+\infty,$$

and if there exists $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $g'(z_1)<0$ and $g'(z_2)<0$, then there exist $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3\in I$ ($\widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3$) such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0,\quad \frac{dg}{dz}(\widetilde{z}_2)>0,\quad \frac{dg}{dz}(\widetilde{z}_3)<0.$$

4.1 Proof of Theorem 3.1(a)

Proof. $\Leftarrow$) First, we prove the sufficiency. Without loss of generality, we assume that

$$\sum_{i\in S_1} a_i>\min_{i\in S_4}\{a_i\}. \tag{4.14}$$

(If $\sum_{i\in S_2} a_i > \min_{i\in S_3}\{a_i\}$, one can similarly prove the network G admits multistability.) By Lemmas 4.1 and 4.2(i), we only need to find $\{d_i\}_{i=1}^s \subset \mathbb{R}$ such that

$$\lim_{z\to\mathcal{L}^+} g(z)=+\infty, \quad \lim_{z\to\mathcal{R}^-} g(z)=-\infty, \quad \lim_{z\to\mathcal{L}^+} \frac{dg}{dz}(z)=-\infty, \quad \lim_{z\to\mathcal{R}^-} \frac{dg}{dz}(z)=-\infty,$$

and there exists $\widetilde{z}\in I$ satisfying $g'(\widetilde{z})>0$. First, for any $i\in S_1$, we let $d_i=\sigma_1$. Similarly, we let $d_i=\sigma_2$ for any $i\in S_2$, and we let $d_i=\sigma_3$ for any $i\in S_3$. Assume that $\min_{i\in S_4}\{a_i\}=a_{i_0}$, where $i_0\in S_4$. And for any $i\in S_4\backslash\{i_0\}$, we also make all d_i's the same, i.e. we let $d_i=\sigma_4$ for any $i\in S_4\backslash\{i_0\}$. Then, by (4.11) and (4.13), we have

$$\begin{aligned} g(z)=&\sum_{i\in S_1} a_i \ln\gamma_i(z+\sigma_1)-\sum_{i\in S_2} a_i \ln\big(-\gamma_i(z-\sigma_2)\big) \\ &+\sum_{i\in S_3} a_i\ln\big(-\gamma_i(z-\sigma_3)\big)-a_{i_0}\ln\gamma_{i_0}(z+d_{i_0})-\sum_{i\in S_4\backslash\{i_0\}} a_i\ln\gamma_i(z+\sigma_4), \\ \frac{dg}{dz}(z)=&\frac{\sum_{i\in S_1}a_i}{z+\sigma_1}+\frac{\sum_{i\in S_2}a_i}{-z+\sigma_2}-\frac{\sum_{i\in S_3}a_i}{-z+\sigma_3}-\frac{a_{i_0}}{z+d_{i_0}}-\frac{\sum_{i\in S_4\backslash\{i_0\}}a_i}{z+\sigma_4}. \end{aligned} \tag{4.15}$$

So,

$$\frac{dg}{dz}(0)=\frac{\sum_{i\in S_1}a_i}{\sigma_1}+\frac{\sum_{i\in S_2}a_i}{\sigma_2}-\frac{\sum_{i\in S_3}a_i}{\sigma_3}-\frac{a_{i_0}}{d_{i_0}}-\frac{\sum_{i\in S_4\backslash\{i_0\}}a_i}{\sigma_4}. \tag{4.16}$$

Below, we will choose concrete values for $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ and d_{i_0} such that $g'(0)>0$. We let

$$\sigma_1=\frac{\sum_{i\in S_1}a_i+a_{i_0}}{2a_{i_0}}, \quad \sigma_3=\frac{2\sum_{i\in S_3}a_i}{c_0}, \quad \sigma_4=\max\left\{\frac{2\sum_{i\in S_4\backslash\{i_0\}}a_i}{c_0},2\right\}, \quad d_{i_0}=1, \tag{4.17}$$

where

$$c_0:=\frac{\sum_{i\in S_1}a_i}{\sigma_1}-\frac{a_{i_0}}{d_{i_0}}=\frac{2a_{i_0}\sum_{i\in S_1}a_i}{\sum_{i\in S_1}a_i+a_{i_0}}-a_{i_0}. \tag{4.18}$$

Notice that by (4.14), we have $\sum_{i\in S_1}a_i>a_{i_0}$. So, by (4.17), we have $\sigma_1>d_{i_0}$ and by (4.18), we have

$$c_0=a_{i_0}\sum_{i\in S_1}a_i+a_{i_0}\left(\sum_{i\in S_1}a_i-a_{i_0}\right)>0. \tag{4.19}$$

Hence, $\sigma_3>0$ (recall that by (3.2) and (3.3), $a_i>0$ for any $i\notin S_5$). Obviously, we can choose σ_2 such that $\sigma_2>\sigma_3$. Notice that $\sum_{i\in S_2}a_i/\sigma_2>0$ (i.e. the second term of $g'(0)$ in (4.16) is positive). By (4.17), we have $c_0/2=\sum_{i\in S_3}a_i/\sigma_3$, and $c_0/2\geqslant\sum_{i\in S_4\backslash\{i_0\}}a_i/\sigma_4$. So, by (4.16), we have

$$\frac{dg}{dz}(0)>\frac{\sum_{i\in S_1}a_i}{\sigma_1}-\frac{a_{i_0}}{d_{i_0}}-\frac{\sum_{i\in S_3}a_i}{\sigma_3}-\frac{\sum_{i\in S_4\backslash\{i_0\}}a_i}{\sigma_4}\geqslant c_0-\frac{c_0}{2}-\frac{c_0}{2}=0. \tag{4.20}$$

Notice that for the interval $I=(\mathcal{L},\mathcal{R})$ defined in (4.12), we have

$$\begin{aligned} &\mathcal{L}=\max\{-\sigma_1,-d_{i_0},-\sigma_4\}=-d_{i_0}<0, \\ &\mathcal{R}=\min\{\sigma_2,\sigma_3\}=\sigma_3>0. \end{aligned}$$

So, we have $0\in I$. By (4.15), when $z\to\mathcal{R}^-$, we have $g'(z)\to-\infty, g(z)\to-\infty$, and when $z\to\mathcal{L}^+$, we have $g'(z)\to-\infty, g(z)\to+\infty$. Hence, by (4.20) and by Lemmas 4.1 and 4.2(i), the network G admits multistablity.

$\Rightarrow$) Next, we prove the necessity. Assume that the network G admits multistability. By Lemma 4.1, G admits multistability iff there exist $\{d_i\}_{i=1}^s\subset\mathbb{R}$ and $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in the interval $I=(\mathcal{L},\mathcal{R})$ defined in (4.12) satisfying $g'(z_1)<0$ and $g'(z_2)<0$, where these d_i's are distinct from each other. Below, we prove the conclusion by deducing a contradiction. We assume that

$$\sum_{i\in S_1}a_i\leqslant\min_{i\in S_4}\{a_i\},\quad \sum_{i\in S_2}a_i\leqslant\min_{i\in S_3}\{a_i\}.$$

Equivalently, we have

$$\sum_{i\in S_1}a_i\leqslant a_j,\quad \forall j\in S_4, \tag{4.21}$$

$$\sum_{i\in S_2}a_i\leqslant a_j,\quad \forall j\in S_3. \tag{4.22}$$

By (4.12), there exists $m\in S_1\cup S_4$ such that $\mathcal{L}=-d_m$ and there exists $n\in S_2\cup S_3$ such that $\mathcal{R}=d_n$. Below, we deduce the contradiction for four different cases.

Case 1. Assume that there exist $m\in S_1$ and $n\in S_3$ such that

$$\mathcal{L}=-d_m,\quad \mathcal{R}=d_n. \tag{4.23}$$

By (4.11) and (4.13), we have

$$\lim_{z\to-d_m^+}g(z)=-\infty,\quad \lim_{z\to d_n^-}g(z)=-\infty,\quad \lim_{z\to-d_m^+}\frac{dg}{dz}(z)=+\infty,\quad \lim_{z\to d_n^-}\frac{dg}{dz}(z)=-\infty.$$

So, by Lemmas 4.1 and 4.2(ii), there exists $z_0\in I=(-d_m,d_n)$ such that

$$\frac{dg}{dz}(z_0)=\sum_{i\in S_1}\frac{a_i}{z_0+d_i}+\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}-\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}-\sum_{i\in S_4}\frac{a_i}{z_0+d_i}=0, \tag{4.24}$$

$$\frac{d^2g}{dz^2}(z_0)=-\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}+\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}-\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}+\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\geqslant 0. \tag{4.25}$$

Below, we show that if (4.24) and (4.25) hold simultaneously, then there will be a contradiction. By (4.12) and (4.23), we have

$$\mathcal{R}=d_n<d_i,\quad \forall i\in S_2. \tag{4.26}$$

So,

$$\frac{a_i}{-z_0+d_i}<\frac{a_i}{-z_0+d_n},\quad \forall i\in S_2.$$

Thus, for the second and the third terms in (4.24), we have

$$\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}-\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}<\frac{\sum_{i\in S_2}a_i}{-z_0+d_n}-\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}. \tag{4.27}$$

By the fact that $n\in S_3$ and by (4.22), we have

$$\frac{\sum_{i\in S_2}a_i}{-z_0+d_n}-\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}\leqslant\frac{\sum_{i\in S_2}a_i}{-z_0+d_n}-\frac{a_n}{-z_0+d_n}\leqslant 0. \tag{4.28}$$

So, by (4.27) and (4.28), we have

$$\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}<\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}. \tag{4.29}$$

Then, by (4.24), for the first and the last terms in (4.24), we have

$$\sum_{i\in S_1}\frac{a_i}{z_0+d_i}>\sum_{i\in S_4}\frac{a_i}{z_0+d_i}. \tag{4.30}$$

Similarly, by (4.26) and (4.22), we have

$$\begin{aligned}
&\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}-\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}\\
<&\frac{\sum_{i\in S_2}a_i}{(-z_0+d_n)^2}-\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}\\
\leqslant&\frac{\sum_{i\in S_2}a_i}{(-z_0+d_n)^2}-\frac{a_n}{(-z_0+d_n)^2}\leqslant 0.
\end{aligned}$$

Then, by (4.25), we have

$$\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}>\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}. \tag{4.31}$$

By (4.21), we have

$$\sum_{i\in S_4}\frac{a_i^2}{(z_0+d_i)^2}\geqslant\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i. \tag{4.32}$$

By (4.31), by using the Cauchy-Schwarz inequality and by (4.30), we have

$$\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i>\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i\geqslant\left(\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\right)^2>\left(\sum_{i\in S_4}\frac{a_i}{z_0+d_i}\right)^2.$$

So, by (4.32), we have

$$\sum_{i\in S_4}\frac{a_i^2}{(z_0+d_i)^2}>\left(\sum_{i\in S_4}\frac{a_i}{z_0+d_i}\right)^2,$$

which is impossible since $a_i/(z_0+d_i)>0$.

Case 2. Assume that there exist $m\in S_4$ and $n\in S_2$ such that

$$\mathcal{L}=-d_m,\quad \mathcal{R}=d_n. \tag{4.33}$$

Since Case 2 is symmetric with respect to Case 1, the proof is similar to the proof of Case 1. We omit the details.

Case 3. Assume that there exist $m\in S_4$ and $n\in S_3$ such that

$$\mathcal{L}=-d_m,\quad \mathcal{R}=d_n. \tag{4.34}$$

Then, by (4.12), we have

$$-d_m=\max\{-d_i\}_{i\in S_1\cup S_4},\quad d_n=\min\{d_i\}_{i\in S_2\cup S_3}.$$

So, for any $z\in(-d_m,d_n)$, we have

$$\begin{aligned}&\frac{a_i}{z+d_i}<\frac{a_i}{z+d_m}, && \forall i\in S_1,\\ &\frac{a_i}{-z+d_i}<\frac{a_i}{-z+d_n}, && \forall i\in S_2.\end{aligned} \tag{4.35}$$

Since $m\in S_4$ and $n\in S_3$, by (4.21)-(4.22), we have

$$a_m\geqslant\sum_{i\in S_1}a_i,\quad a_n\geqslant\sum_{i\in S_2}a_i. \tag{4.36}$$

Then, by (4.13), (4.35) and (4.36), we have

$$\begin{aligned}\frac{dg}{dz}(z)&=\sum_{i\in S_1}\frac{a_i}{z+d_i}+\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}-\sum_{i\in S_4}\frac{a_i}{z+d_i}\\ &\leqslant\frac{\sum_{i\in S_1}a_i}{z+d_m}+\frac{\sum_{i\in S_2}a_i}{-z+d_n}-\frac{a_n}{-z+d_n}-\frac{a_m}{z+d_m}\leqslant 0.\end{aligned}$$

So, $g(z)$ is decreasing in I. Thus, $g(z)=0$ has at most one real solution in I. On the other hand, by Lemma 4.1, $g(z)=0$ has at least two real solutions in I, which is a contradiction.

Case 4. Assume that there exist $m\in S_1$ and $n\in S_2$ such that

$$\mathcal{L}=-d_m,\quad \mathcal{R}=d_n. \tag{4.37}$$

By (4.11) and (4.13), we have

$$\lim_{z\to -d_m^+}g(z)=-\infty,\quad \lim_{z\to d_n^-}g(z)=+\infty,\quad \lim_{z\to -d_m^+}\frac{dg}{dz}(z)=+\infty,\quad \lim_{z\to d_n^-}\frac{dg}{dz}(z)=+\infty.$$

So, by Lemmas 4.1 and 4.2(iii), if G admits multistability, then there exists $z_0\in I=(-d_m,d_n)$ such that

$$\frac{dg}{dz}(z_0)>0,\quad \frac{d^2g}{dz^2}(z_0)=0,\quad \frac{d^3g}{dz^3}(z_0)\leqslant 0.$$

So, by (4.13), we have

$$\sum_{i\in S_1}\frac{a_i}{z_0+d_i}+\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}>\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}+\sum_{i\in S_4}\frac{a_i}{z_0+d_i}, \tag{4.38}$$

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}+\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}=\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}+\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}, \tag{4.39}$$

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^3}+\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^3}\leqslant\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^3}+\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}. \tag{4.40}$$

Note that by (4.40), one of the following two equations must hold:

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^3}\leqslant\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}, \tag{4.41}$$

$$\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^3}\leqslant\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^3}. \tag{4.42}$$

Without loss of generality, we assume (4.41) holds (if (4.42) holds, we can prove the conclusion similarly). Below, we prove that (4.38), (4.39) and (4.41) can not hold simultaneously by the following four steps.

Step 1. In this step, we prove that (4.38) and (4.39) imply

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}>\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}. \tag{4.43}$$

We prove the conclusion by deducing a contradiction. Assume that

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\leqslant\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}. \tag{4.44}$$

Then, by Cauchy inequality and by (4.44), we have

$$\left(\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\right)^2\leqslant\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i\leqslant\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i. \tag{4.45}$$

By (4.21), we have

$$\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}a_i\leqslant\sum_{i\in S_4}\frac{a_i^2}{(z_0+d_i)^2}\leqslant\left(\sum_{i\in S_4}\frac{a_i}{z_0+d_i}\right)^2. \tag{4.46}$$

Notice that $a_i/(z_0+d_i)>0$ for any $z_0\in I$. So, by (4.45) and (4.46), we have

$$\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\leqslant\sum_{i\in S_4}\frac{a_i}{z_0+d_i}.$$

Hence, by (4.38), we have

$$\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}>\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}. \tag{4.47}$$

On the other hand, by (4.39) and (4.44), we have

$$\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}\leqslant\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}. \tag{4.48}$$

So, by Cauchy inequality and by (4.48), we have

$$\left(\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}\right)^2\leqslant\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}\cdot\sum_{i\in S_2}a_i\leqslant\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}\cdot\sum_{i\in S_2}a_i. \tag{4.49}$$

By (4.22) and by the fact $a_{i_2}\leqslant a_i$ for any $i\in S_3$, we have

$$\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}\cdot\sum_{i\in S_2}a_i\leqslant\sum_{i\in S_3}\frac{a_i^2}{(-z_0+d_i)^2}\leqslant\left(\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}\right)^2. \tag{4.50}$$

So, by (4.49) and (4.50), we have

$$\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}\leqslant\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}.$$

This is a contradiction to (4.47). Therefore, the inequality (4.43) must hold.

Step 2. In this step, we show that by (4.41) and (4.43), we can construct $d_0,a_0\in\mathbb{R}$ such that

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}>\frac{a_0}{(z_0+d_0)^2}, \tag{4.51}$$

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^3}\leqslant\frac{a_0}{(z_0+d_0)^3}. \tag{4.52}$$

Let

$$d_0:=\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)\Big/\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}\right)-z_0, \tag{4.53}$$

$$a_0:=\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^3\Big/\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}\right)^2. \tag{4.54}$$

It is straightforward to check that

$$\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}=\frac{a_0}{(z_0+d_0)^2},\quad \sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}=\frac{a_0}{(z_0+d_0)^3}. \tag{4.55}$$

So, by (4.41) and (4.43), we have (4.51) and (4.52) hold.

Step 3. In this step, we prove that

$$a_0 \geqslant \sum_{i\in S_1} a_i. \tag{4.56}$$

By (4.54), we only need to show that

$$\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^3 \geqslant \left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}\right)^2 \cdot \sum_{i\in S_1} a_i. \tag{4.57}$$

Notice that by Cauchy inequality, we have

$$\begin{aligned}
&\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^3 \cdot \sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^4} \\
=&\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^2 \cdot \sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2} \cdot \sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^4} \\
\geqslant&\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^2 \cdot \left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^3}\right)^2.
\end{aligned} \tag{4.58}$$

By (4.21), we have

$$\left(\sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^2}\right)^2 \geqslant \sum_{i\in S_4}\frac{a_i^2}{(z_0+d_i)^4} \geqslant \sum_{i\in S_4}\frac{a_i}{(z_0+d_i)^4} \cdot \sum_{i\in S_1} a_i. \tag{4.59}$$

So, by (4.58) and (4.59), one can see that (4.57) holds.

Step 4. In this step, we show that (4.51) and (4.52) imply two contradictory inequalities (4.64) and (4.65). By Cauchy inequality and by (4.52), we have

$$\left(\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\right)^2 \leqslant \sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^3} \cdot \sum_{i\in S_1}\frac{a_i}{z_0+d_i} \leqslant \frac{a_0}{(z_0+d_0)^3} \cdot \sum_{i\in S_1}\frac{a_i}{z_0+d_i}. \tag{4.60}$$

We multiply the left-hand side and the right-hand side of (4.60) by $\sum_{i\in S_1} a_i$, and we get

$$\left(\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\right)^2 \cdot \sum_{i\in S_1} a_i \leqslant \frac{a_0}{(z_0+d_0)^3} \cdot \sum_{i\in S_1}\frac{a_i}{z_0+d_i} \cdot \sum_{i\in S_1} a_i. \tag{4.61}$$

Note that by Cauchy inequality, we have

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2} \cdot \left(\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\right)^2 \leqslant \left(\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\right)^2 \cdot \sum_{i\in S_1} a_i. \tag{4.62}$$

By (4.56), we have $\sum_{i\in S_1} a_i \leqslant a_0$. So,

$$\frac{a_0}{(z_0+d_0)^3}\cdot\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\cdot\sum_{i\in S_1}a_i \leqslant \frac{a_0^2}{(z_0+d_0)^3}\cdot\sum_{i\in S_1}\frac{a_i}{z_0+d_i}. \tag{4.63}$$

Thus, by (4.61)-(4.63), we have

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\cdot\left(\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\right)^2 \leqslant \frac{a_0^2}{(z_0+d_0)^3}\cdot\sum_{i\in S_1}\frac{a_i}{z_0+d_i}.$$

Note that $a_i/(z_0+d_i)>0$ $(i\in S_1)$ for any $z_0\in I$. So, we have

$$\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\cdot\sum_{i\in S_1}\frac{a_i}{z_0+d_i}\leqslant\frac{a_0^2}{(z_0+d_0)^3}.$$

Then, by (4.51), we have

$$\sum_{i\in S_1}\frac{a_i}{z_0+d_i}<\frac{a_0}{z_0+d_0}. \tag{4.64}$$

On the other hand, by (4.51) and (4.60), we have

$$\frac{a_0^2}{(z_0+d_0)^4}<\left(\sum_{i\in S_1}\frac{a_i}{(z_0+d_i)^2}\right)^2\leqslant\frac{a_0}{(z_0+d_0)^3}\cdot\sum_{i\in S_1}\frac{a_i}{z_0+d_i}.$$

Note that by (4.53) and (4.54), we have $z_0+d_0>0$ and $a_0>0$. So,

$$\frac{a_0}{z_0+d_0}<\sum_{i\in S_1}\frac{a_i}{z_0+d_i}, \tag{4.65}$$

which is a contradiction to (4.64). So far, we have deduced the contradiction for the last case, and we complete the proof. □

4.2 Proof of Theorem 3.1(b)

In this case, there are exactly three of the four sets S_1,S_2,S_3,S_4 are non-empty. Below, we successively prove Theorem 3.1(b)(1)-(4).

4.2.1 Proof of Theorem 3.1(b)(1)

Proof. According to the hypothesis of Theorem 3.1(b)(1), we assume that S_1,S_3 and S_4 are non-empty. By (4.11), we have

$$g(z)=\sum_{i\in S_1}a_i\ln(z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i), \tag{4.66}$$

$$\frac{dg}{dz}(z)=\sum_{i\in S_1}\frac{a_i}{z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}-\sum_{i\in S_4}\frac{a_i}{z+d_i}. \tag{4.67}$$

$\Rightarrow$) First, we prove the sufficiency. By [27, Theorem 3.4], if the network G admits multistability, then $cap_{pos}(G)\geq 3$. By [26, Theorem 6.1(c)], if $cap_{pos}(G)\geq 3$, then we have $\sum_{i\in S_1}a_i>\min_{i\in S_4}\{a_i\}$.

$\Leftarrow$) Next, we prove the necessity. Assume that

$$\sum_{i\in S_1}a_i>\min_{i\in S_4}\{a_i\}. \tag{4.68}$$

The goal is to prove that G admits multistability. Assume that $a_p=\min_{i\in S_4}\{a_i\}$, where $p\in S_4$. First, we let $d_p=0$. Then, for any $i\in S_1$, we let $d_i=d$, and for any $i\in S_3$, we let $d_i=1$. For any $S_4\backslash\{p\}$, we also make all d_i's the same, i.e. we let $d_i=e$ for any $i\in S_4\backslash\{p\}$. Notice that d and e are two positive parameters, and we will choose proper values for them later. By (4.67), we have

$$\frac{dg}{dz}(z)=\frac{\sum_{i\in S_1}a_i}{z+d}-\frac{\sum_{i\in S_3}a_i}{-z+1}-\frac{a_p}{z}-\frac{\sum_{i\in S_4\backslash\{p\}}a_i}{z+e}. \tag{4.69}$$

Note that the interval I defined in (4.12) is $(0,1)$. By (4.66) and (4.67), we have

$$\lim_{z\to 0^+}g(z)=+\infty,\quad \lim_{z\to 1^-}g(z)=-\infty,\quad \lim_{z\to 0^+}\frac{dg}{dz}(z)=-\infty,\quad \lim_{z\to 1^-}\frac{dg}{dz}(z)=-\infty.$$

By Lemmas 4.1 and 4.2(i), we only need to choose proper positive numbers d and e such that there exists $\widetilde{z}\in I$ satisfying $g'(\widetilde{z})>0$. Below, we complete the proof by the following two steps.

Step 1. Let

$$h(z):=\frac{\sum_{i\in S_1}a_i}{z+d}-\frac{\sum_{i\in S_3}a_i}{-z+1}-\frac{a_p}{z}. \tag{4.70}$$

Notice that by (4.69),

$$h(z)=\frac{dg}{dz}(z)+\frac{\sum_{i\in S_4\backslash\{p\}}a_i}{z+e}.$$

In this step, we show that we can choose $d>0$ such that there exists $\widetilde{z}\in I$ satisfying $h(\widetilde{z})>0$. We solve d from $h(z)>0$ by (4.70), and we get

$$d<\frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \tag{4.71}$$

where

$$\mathcal{D}(z):=\frac{\sum_{i\in S_3}a_i}{-z+1}+\frac{a_p}{z},\quad \mathcal{N}(z):=\sum_{i\in S_1}a_i-\frac{z}{-z+1}\sum_{i\in S_3}a_i-a_p.$$

Notice that by (4.68), we have $\sum_{i\in S_1}a_i>a_p$, and so,

$$\lim_{z\to 0}\mathcal{N}(z)>0. \tag{4.72}$$

So, there exists $\widetilde{z}\in I=(0,1)$ such that the $\mathcal{N}(\widetilde{z})>0$. Notice that for any $z\in I=(0,1)$, we have $\mathcal{D}(z)>0$. Therefore, $\mathcal{N}(\widetilde{z})/\mathcal{D}(\widetilde{z})>0$. Then, we can choose an appropriate positive number d such that $d<\mathcal{N}(\widetilde{z})/\mathcal{D}(\widetilde{z})$, i.e. $h(\widetilde{z})>0$.

Step 2. In this step, we prove that we can choose $e>0$ such that $g'(\widetilde{z})>0$. In fact, let

$$e=2\frac{\sum_{i\in S_4\setminus\{p\}}a_i}{h(\widetilde{z})}.$$

Then,

$$\frac{dg}{dz}(\widetilde{z})=h(\widetilde{z})-\frac{\sum_{i\in S_4\setminus\{p\}}a_i}{\widetilde{z}+e}>h(\widetilde{z})-\frac{\sum_{i\in S_4\setminus\{p\}}a_i}{e}=h(\widetilde{z})-\frac{h(\widetilde{z})}{2}>0.$$

The proof is complete. □

4.2.2 Proof of Theorem 3.1(b)(2)

Proof. According to the hypothesis of Theorem 3.1(b)(2), we assume that S_2,S_3 and S_4 are non-empty. By (4.11), we have

$$g(z)=-\sum_{i\in S_2}a_i\ln(-z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i).$$

Define

$$\widetilde{g}(z)=-g(-z)=\sum_{i\in S_2}a_i\ln(z+d_i)-\sum_{i\in S_3}a_i\ln(z+d_i)+\sum_{i\in S_4}a_i\ln(-z+d_i).$$

Notice that $\widetilde{g}\,'(z)=g'(-z)$. Let $I^*:=\{-z|z\in I\}$. Then, there exist $z_1,z_2\in I$ such that $g(z_i)=0$, and $g'(z_i)<0$ $(i=1,2)$ if and only if there exist $z_1^*,z_2^*\in I^*$ such that $\widetilde{g}(z_i^*)=0$, and $\widetilde{g}\,'(z_i^*)<0$ $(i=1,2)$. Note that by the proof of Theorem 3.1(b)(1), there exist $z_1^*,z_2^*\in I^*$ such that $\widetilde{g}(z_i^*)=0$, and $\widetilde{g}\,'(z_i^*)<0$ $(i=1,2)$ if and only if $\sum_{i\in S_2}a_i>\min_{i\in S_3}\{a_i\}$. So, by Lemma 4.1, G admits multistability if and only if $\sum_{i\in S_2}a_i>\min_{i\in S_3}\{a_i\}$. □

4.2.3 Proof of Theorem 3.1(b)(3)

First, we present some useful lemmas (Lemmas 4.3-4.6). Since the proofs of these lemmas are elementary, we put them in the supplementary materials[§].

Lemma 4.3. *For any $\beta_1,\beta_2,e_1,e_2,x_1,x_2,x_3\in R$ satisfying*

$$\beta_1>0,\quad \beta_2>0, \tag{4.73}$$

$$x_i+e_j>0\quad (i=1,2,3,\ j=1,2), \tag{4.74}$$

$$e_1\neq e_2, \tag{4.75}$$

[§]https://github.com/65536-1024/one-dim *where exist $a,b,c\in R$ such that*

$$\frac{\beta_1}{x_i+e_1}+\frac{\beta_2}{x_i+e_2}=a+\frac{c}{x_i+b} \quad (i=1,2,3), \tag{4.76}$$

where

$$a>0, \quad b>\min\{e_1,e_2\}, \quad \min\{\beta_1,\beta_2\}<c<\beta_1+\beta_2. \tag{4.77}$$

Lemma 4.4. *Let $G(z):=\sum_{i=1}^n a_i/(z+d_i)$, where $d_i\in R$, and $a_i>0$. Let $M:=\min_{i\in\{1,\dots,n\}}\{d_i\}$. Then, for any three different numbers z_1,z_2,z_3 satisfying $z_j>-M$ $(j=1,2,3)$, there exist A,D, $\theta\in R$ such that*

$$G(z_j)=\frac{A}{z_j+D}+\theta \quad (j=1,2,3), \tag{4.78}$$

where

$$\min_{i\in\{1,\dots,n\}}\{a_i\}\le A\le\sum_{i=1}^n a_i, D\ge M, \quad \theta\ge 0. \tag{4.79}$$

Lemma 4.5. *For any two sequences $\{a_i\}_{i=0}^n$, $\{e_i\}_{i=0}^n$ satisfying*

$$a_i>1 \quad (i=1,\dots,n), \quad e_i>1 \quad (i=1,\dots,n), \quad a_0>\sum_{i=1}^n a_i, \quad e_0>0, \tag{4.80}$$

the following inequalities can not hold simultaneously:

$$\frac{a_0}{e_0}\le\sum_{i=1}^n\frac{a_i}{e_i}-1, \tag{4.81}$$

$$\frac{a_0}{e_0^2}\ge\sum_{i=1}^n\frac{a_i}{e_i^2}-1, \tag{4.82}$$

$$\frac{a_0}{e_0^3}\le\sum_{i=1}^n\frac{a_i}{e_i^3}-1. \tag{4.83}$$

Lemma 4.6. *Define*

$$\begin{aligned} E(x,y,z):=&(1-x)^2(yz-x)(y-z)^2+(1-y)^2(xz-y)(x-z)^2\\ &+(1-z)^2(xy-z)(x-y)^2. \end{aligned} \tag{4.84}$$

Then, for any $x,y,z\in(0,1)$, we have $E(x,y,z)<0$.

Now, we are prepared to prove Theorem 3.1(b)(3).

Proof. According to the hypothesis of Theorem 3.1(b)(3), we assume that S_1,S_2 and S_3 are non-empty. By (4.11), we have

$$g(z)=\sum_{i\in S_1}a_i\ln(z+d_i)-\sum_{i\in S_2}a_i\ln(-z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i), \tag{4.85}$$

$$\frac{dg}{dz}(z)=\sum_{i\in S_1}\frac{a_i}{z+d_i}+\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}. \tag{4.86}$$

Notice that by (4.86), we have

$$\frac{d^2g}{dz^2}(z)=-\sum_{i\in S_1}\frac{a_i}{(z+d_i)^2}+\sum_{i\in S_2}\frac{a_i}{(-z+d_i)^2}-\sum_{i\in S_3}\frac{a_i}{(-z+d_i)^2}. \tag{4.87}$$

$\Rightarrow$) First, we prove the sufficiency. Assume that there exists a subset S_2^* of S_2, such that

$$\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i>\min_{i\in S_3}\{a_i\}. \tag{4.88}$$

The goal is to prove that G admits multistability. Assume that $a_p=\min_{i\in S_3}\{a_i\}$, where $p\in S_3$. First, we let $d_p=1$, and for any $i\in S_1$, we let $d_i=0$. Then, for any $i\in S_2^*$, we let $d_i=w_1$ and for any $i\in S_2\backslash S_2^*$, we let $d_i=w_2$ for any $i\in S_2\backslash S_2^*$. Similarly, for any $i\in S_3\backslash\{p\}$, we also make all d_i's the same, i.e. we let $d_i=w_3$ for any $i\in S_3\backslash\{p\}$. Here, w_i is a new real parameter, and we assume that $w_i>1$ $(i=1,2,3)$. Then, by (4.86), we have

$$\frac{dg}{dz}(z)=\frac{\sum_{i\in S_1}a_i}{z}+\frac{\sum_{i\in S_2^*}a_i}{-z+w_1}+\frac{\sum_{i\in S_2\backslash S_2^*}a_i}{-z+w_2}-\frac{a_p}{-z+1}-\frac{\sum_{i\in S_3\backslash\{p\}}a_i}{-z+w_3}.$$

Note that the interval I defined in (4.12) is now $(0,1)$. By (4.85) and (4.86), we have

$$\lim_{z\to 0^+}g(z)=-\infty,\quad \lim_{z\to 1^-}g(z)=-\infty,\quad \lim_{z\to 0^+}\frac{dg}{dz}(z)=+\infty,\quad \lim_{z\to 1^-}\frac{dg}{dz}(z)=-\infty.$$

So, by Lemmas 4.1 and 4.2(ii), we only need to choose proper values for w_1,w_2, and w_3 such that there exist $\widetilde{z}_1,\widetilde{z}_2\in I$ $(\widetilde{z}_1<\widetilde{z}_2)$ satisfying $g'(\widetilde{z}_1)<0$ and $g'(\widetilde{z}_2)>0$. Let

$$h(z):=\frac{\sum_{i\in S_1}a_i}{z}+\frac{\sum_{i\in S_2^*}a_i}{-z+w_1}-\frac{a_p}{-z+1}-\frac{\sum_{i\in S_3\backslash\{p\}}a_i}{-z+w_3}.$$

We complete the proof by the following three steps.

Step 1. In this step, we prove that there exist $w_3>1$ and $\widetilde{z}_1\in I$ such that for any $w_1>1$, we have $h(\widetilde{z}_1)<0$. In fact, for any $w_1>1$, we have

$$h(z)<\frac{\sum_{i\in S_1}a_i}{z}+\frac{\sum_{i\in S_2^*}a_i-a_p}{-z+1}-\frac{\sum_{i\in S_3\backslash\{p\}}a_i}{-z+w_3}. \tag{4.89}$$

Let the right-hand side of (4.89) be $H(z)$. We solve d from $H(z)<0$, and we get

$$w_3<\frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \tag{4.90}$$

where

$$\mathcal{N}(z):=\sum_{i\in S_3\backslash\{p\}}a_i+\sum_{i\in S_1}a_i+\left(\sum_{i\in S_2^*}a_i-a_p\right)\frac{z}{-z+1},$$

$$\mathcal{D}(z):=\sum_{i\in S_1}a_i\frac{1}{z}+\left(\sum_{i\in S_2^*}a_i-a_p\right)\frac{1}{-z+1}.$$

Note that

$$\mathcal{N}(z)-\mathcal{D}(z)=\sum_{i\in S_3}a_i-\sum_{i\in S_2^*}a_i+\sum_{i\in S_1}a_i\left(1-\frac{1}{z}\right).$$

By (4.88), we have $\lim_{z\to 1}(\mathcal{N}(z)-\mathcal{D}(z))=\sum_{i\in S_3}a_i-\sum_{i\in S_2^*}a_i>0$. So, there exists $\widetilde{z}_1\in(0,1)$ such that $\mathcal{N}(\widetilde{z}_1)-\mathcal{D}(\widetilde{z}_1)>0$. Note also by (4.88), we have for any $z\in(0,1)$, $\mathcal{N}(z)>0$ and $\mathcal{D}(z)>0$. Hence,

$$\frac{\mathcal{N}(\widetilde{z}_1)}{\mathcal{D}(\widetilde{z}_1)}>1.$$

By (4.90), there exists $w_3>1$ such that $H(\widetilde{z}_1)<0$. Recall that for any $w_1>1$, we have (4.89), i.e. $h(z)<H(z)$. So, for any $w_1>1$, we have $h(\widetilde{z}_1)<0$.

Step 2. In this step, we prove that there exist $w_1>1$ and $\widetilde{z}_2\in(\widetilde{z}_1,1)$ such that $h(\widetilde{z}_2)>0$. In fact, we can solve w_1 from $h(z)>0$, and we get

$$w_1<\frac{\widetilde{\mathcal{N}}(z)}{\widetilde{\mathcal{D}}(z)}, \tag{4.91}$$

where

$$\begin{aligned}\widetilde{\mathcal{D}}(z)&:=-\sum_{i\in S_1}a_i\frac{1}{z}+\frac{a_p}{-z+1}+\left(\sum_{i\in S_3\backslash\{p\}}a_i\right)\frac{1}{-z+w_3},\\ \widetilde{\mathcal{N}}(z)&:=\sum_{i\in S_2^*}a_i-\sum_{i\in S_1}a_i+a_p\frac{z}{-z+1}+\left(\sum_{i\in S_3\backslash\{p\}}a_i\right)\frac{z}{-z+w_3}.\end{aligned}$$

Since

$$\lim_{z\to 1^-}\widetilde{\mathcal{D}}(z)=+\infty,\quad \lim_{z\to 1^-}\widetilde{\mathcal{N}}(z)=+\infty, \tag{4.92}$$

there exists $z^*\in(0,1)$ such that for any $z\in(z^*,1)$,

$$\widetilde{\mathcal{D}}(z)>0,\quad \widetilde{\mathcal{N}}(z)>0. \tag{4.93}$$

Note that

$$\widetilde{\mathcal{N}}(z)-\widetilde{\mathcal{D}}(z)=\sum_{i\in S_2^*}a_i-a_p+\sum_{i\in S_1}a_i\left(\frac{1}{z}-1\right)+\left(\sum_{i\in S_3\backslash\{p\}}a_i\right)\frac{z-1}{-z+w_3}.$$

Since $w_3>1$, we have

$$\lim_{z\to 1}\left(\widetilde{\mathcal{N}}(z)-\widetilde{\mathcal{D}}(z)\right)=\sum_{i\in S_2^*}a_i-a_p.$$

By (4.88), the above limit is positive. Therefore, we can choose $\widetilde{z}_2\in(\max\{\widetilde{z}_1,z^*\},1)$ such that

$$\frac{\widetilde{\mathcal{N}}(\widetilde{z}_2)}{\widetilde{\mathcal{D}}(\widetilde{z}_2)}>1.$$

So, by (4.91), we can choose appropriate $w_1>1$ such that $h(\widetilde{z}_2)>0$.

Step 3. In this step, we prove that there exists $w_2>1$ such that $g'(\widetilde{z}_1)<0$ and $g'(\widetilde{z}_2)>0$. In fact, we can choose

$$w_2=\max\left\{\frac{2\sum_{i\in S_2\setminus S_2^*}a_i}{-h(\widetilde{z}_1)}+\widetilde{z}_1,2\right\}.$$

Therefore, we have

$$\begin{aligned}\frac{dg}{dz}(\widetilde{z}_1)&=h(\widetilde{z}_1)+\frac{\sum_{i\in S_2\setminus S_2^*}a_i}{-\widetilde{z}_1+w_2}\le\frac{h(\widetilde{z}_1)}{2}<0,\\ \frac{dg}{dz}(\widetilde{z}_2)&=h(\widetilde{z}_2)+\frac{\sum_{i\in S_2\setminus S_2^*}a_i}{-\widetilde{z}_2+w_2}>h(\widetilde{z}_2)>0.\end{aligned}$$

$\Leftarrow$) Next, we prove the necessity. Our goal is to prove that if G admits multistability, then there exists a subset S_2^* of S_2 such that

$$\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i>\min_{i\in S_3}\{a_i\}. \tag{4.94}$$

Assume that $|S_i|=s_i$ $(i=1,2,3)$, and assume that $S_1=\{1,\ldots,s_1\}$, $S_2=\{s_1+1,\ldots,s_1+s_2\}$, and $S_3=\{s_1+s_2+1,\ldots,s_1+s_2+s_3\}$. Below, we prove the conclusion by deducing a contradiction. Note that if there does not exist a subset S_2^* of S_2 such that $\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i>\min_{i\in S_3}\{a_i\}$, then we have the following three cases.

Case 1. $s_3=1$.

Case 2. $s_3\ge 2$ and for any $i\in S_2$, we have $\sum_{i\in S_3}a_i\le a_i$.

Case 3. Assume that $a_{s_1+1}\le a_{s_1+2}\le\cdots\le a_{s_1+s_2}$. There exists $k\in\{1,\ldots,s_2\}$ such that

$$\sum_{i=s_1+1}^{s_1+k}a_i\le\min_{i\in S_3}\{a_i\}<\sum_{i\in S_3}a_i\le a_{s_1+k+1}\le\cdots\le a_{s_1+s_2}.$$

Below, we will prove the conclusion by discussing the three cases. By Lemma 4.1, if G admits multistability, then there exist $\{d_i\}_{i=1}^{s}\subset\mathbb{R}$ and $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in the interval $I=(\mathcal{L},\mathcal{R})$ defined in (4.12) satisfying $g'(z_1)<0$ and $g'(z_2)<0$, where these d_i's are distinct from each other.

Case 1. Assume that $s_3=1$. Then, $S_3=\{s_1+s_2+1\}$. Suppose $d_1<d_2<\cdots<d_{s_1}$, and $d_{s_1+1}<d_{s_1+2}<\cdots<d_{s_1+s_2}$.

Case 1.1. If $d_{s_1+s_2+1}<d_{s_1+1}$, then the interval I defined in (4.12) is $(-d_1,d_{s_1+s_2+1})$. Notice that by (4.86), for any $i\in\{s_1+1,\ldots,s_1+s_2-1\}$, $\lim_{z\to d_i^+}g'(z)=-\infty$ and $\lim_{z\to d_{i+1}^-}g'(z)=+\infty$. Note also that $g'(z)$ is continuous in (d_i,d_{i+1}). So, there exists $z_i\in(d_i,d_{i+1})$ such that $g'(z_i)=0$. Hence, $g'(z)=0$ has at least s_2-1 solutions in $(d_{s_1+1},+\infty)$. Similarly, notice that by (4.86), for any $i\in\{1,\ldots,s_1-1\}$, $\lim_{z\to -d_i^-}g'(z)=-\infty$ and $\lim_{z\to -d_{i+1}^+}g'(z)=+\infty$. So, there exists $z_i\in(-d_{i+1},-d_i)$ such that $g'(z_i)=0$.

Hence, $g'(z)=0$ has at least s_1-1 solutions in $(-\infty,-d_1)$. Since the numerator of $g'(z)$ is a polynomial with degree s_1+s_2, $g'(z)=0$ has no more than s_1+s_2 real solutions in $(-\infty,+\infty)$. Hence, there are no more than 2 solutions in $I=(-d_1,d_{s_1+s_2+1})$. On the other hand, by (4.85) and (4.86), notice that

$$\lim_{z\to -d_1^+} g(z)=-\infty, \quad \lim_{z\to d_{s_1+s_2+1}^-} g(z)=-\infty, \quad \lim_{z\to -d_1^+}\frac{dg}{dz}(z)=+\infty, \quad \lim_{z\to d_{s_1+s_2+1}^-}\frac{dg}{dz}(z)=-\infty.$$

By Lemmas 4.1 and 4.2(ii), if G admits multistability, then there exist $\widetilde{z}_1,\widetilde{z}_2\in I$ $(\widetilde{z}_1<\widetilde{z}_2)$ such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0, \quad \frac{dg}{dz}(\widetilde{z}_2)>0.$$

Since $\lim_{z\to -d_1^+} g'(z)=+\infty$ and $\lim_{z\to d_{s_1+s_2+1}^-} g'(z)=-\infty$, $g'(z)=0$ has at least 3 solutions in I, which is a contradiction.

Case 1.2. If $d_{s_1+s_2+1}>d_{s_1+1}$, then the interval I defined in (4.12) is $(-d_1,d_{s_1+1})$. Notice that by (4.86), for any $i\in\{s_1+1,\dots,s_1+s_2-1\}$, we have $\lim_{z\to d_i^+} g'(z)=-\infty$ and $\lim_{z\to d_{i+1}^-} g'(z)=+\infty$. So, for any $i\in\{s_1+1,\dots,s_1+s_2-1\}$ satisfying $d_{s_1+s_2+1}\notin(d_i,d_{i+1})$, there exists $z_i\in(d_i,d_{i+1})$ such that $g'(z_i)=0$. Note that $d_{s_1+s_2+1}$ is located in at most one of the s_2-1 intervals $(d_i,d_{i+1})(i\in\{s_1+1,\dots,s_1+s_2-1\})$. Hence, $g'(z)=0$ has at least s_2-2 real solutions in $(d_{s_1+1},+\infty)$. Similarly, notice that by (4.86), for any $i\in\{1,\dots,s_1-1\}$, $\lim_{z\to -d_i^-} g'(z)=-\infty$ and $\lim_{z\to -d_{i+1}^+} g'(z)=+\infty$. So, there exists $z_i\in(-d_{i+1},-d_i)$ such that $g'(z_i)=0$. Hence, $g'(z)=0$ has at least s_1-1 real solutions in $(-\infty,-d_1)$. Since the numerator of $g'(z)$ is a polynomial with degree s_1+s_2, $g'(z)=0$ has no more than s_1+s_2 real solutions in $(-\infty,+\infty)$. Hence, $g'(z)=0$ has no more than 3 real solutions in I. On the other hand, by (4.85) and (4.86), notice that

$$\lim_{z\to -d_1^+} g(z)=-\infty, \quad \lim_{z\to d_{s_1+1}^-} g(z)=+\infty, \quad \lim_{z\to -d_1^+}\frac{dg}{dz}(z)=+\infty, \quad \lim_{z\to d_{s_1+1}^-}\frac{dg}{dz}(z)=+\infty.$$

By Lemmas 4.1 and 4.2(iii), if G admits multistability, then there exist $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3\in I$ $(\widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3)$ such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0, \quad \frac{dg}{dz}(\widetilde{z}_2)>0, \quad \frac{dg}{dz}(\widetilde{z}_3)<0.$$

Since $\lim_{z\to -d_1^+} g'(z)=+\infty$ and $\lim_{z\to d_{s_1+1}^-} g'(z)=+\infty$, $g'(z)=0$ has at least 4 solutions in I, which is a contradiction.

Case 2. Recall that the hypothesis of this case is that $s_3\geq 2$ and for any $i\in S_2$, we have $\sum_{i\in S_3} a_i\leq a_i$. Notice that the interval I defined in (4.12) is

$$I=(\mathcal{L},\mathcal{R}), \tag{4.95}$$

where

$$\mathcal{L} = -\min_{i\in S_1}\{d_i\}, \tag{4.96}$$

$$\mathcal{R} = \min\{d_i\}_{i\in S_2\cup S_3}. \tag{4.97}$$

Step 1. First, we prove that if G admits multistability, then there exist $\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3 \in I$ $(\widetilde{z}_1 < \widetilde{z}_2 < \widetilde{z}_3)$, such that

$$\frac{dg}{dz}(\widetilde{z}_1) < 0, \quad \frac{dg}{dz}(\widetilde{z}_2) > 0, \quad \frac{dg}{dz}(\widetilde{z}_3) < 0. \tag{4.98}$$

If $\min_{i\in S_2}\{d_i\} < \min_{i\in S_3}\{d_i\}$, we have $I = (-\min_{i\in S_1}\{d_i\}, \min_{i\in S_2}\{d_i\})$. By (4.85) and (4.86), we have

$$\lim_{z\to\mathcal{L}^+} g(z) = -\infty, \quad \lim_{z\to\mathcal{R}^-} g(z) = +\infty, \quad \lim_{z\to\mathcal{L}^+} \frac{dg}{dz}(z) = +\infty, \quad \lim_{z\to\mathcal{R}^-} \frac{dg}{dz}(z) = +\infty.$$

By Lemma 4.2(iii), there exist $\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3 \in I$ $(\widetilde{z}_1 < \widetilde{z}_2 < \widetilde{z}_3)$ such that

$$\frac{dg}{dz}(\widetilde{z}_1) < 0, \quad \frac{dg}{dz}(\widetilde{z}_2) > 0, \quad \frac{dg}{dz}(\widetilde{z}_3) < 0. \tag{4.99}$$

If $\min_{i\in S_3}\{d_i\} < \min_{i\in S_2}\{d_i\}$, we have $I = (-\min_{i\in S_1}\{d_i\}, \min_{i\in S_3}\{d_i\})$. By (4.85) and (4.86), we have

$$\lim_{z\to\mathcal{L}^+} g(z) = -\infty, \quad \lim_{z\to\mathcal{R}^-} g(z) = -\infty, \quad \lim_{z\to\mathcal{L}^+} \frac{dg}{dz}(z) = +\infty, \quad \lim_{z\to\mathcal{R}^-} \frac{dg}{dz}(z) = -\infty.$$

By Lemma 4.2(ii), there exist $\widetilde{z}_1, \widetilde{z}_2 \in I$ $(\widetilde{z}_1 < \widetilde{z}_2)$ such that

$$\frac{dg}{dz}(\widetilde{z}_1) < 0, \quad \frac{dg}{dz}(\widetilde{z}_2) > 0. \tag{4.100}$$

Since $\lim_{z\to\mathcal{R}^-} g'(z) = -\infty$, there exists $\widetilde{z}_3 \in (\widetilde{z}_2, R)$ such that $g'(\widetilde{z}_3) < 0$. So, the conclusion holds.

Step 2. Below, we deduce a contradiction by (4.98). Note that $g'(z)$ is a rational function. So, by (4.98), there exists $z_0 \in (\widetilde{z}_1, \widetilde{z}_2)$ such that

$$\frac{dg}{dz}(z_0) = 0, \quad \frac{d^2g}{dz^2}(z_0) \geq 0. \tag{4.101}$$

By (4.86), (4.87) and (4.101), we have

$$\sum_{i\in S_1} \frac{a_i}{z_0 + d_i} + \sum_{i\in S_2} \frac{a_i}{-z_0 + d_i} - \sum_{i\in S_3} \frac{a_i}{-z_0 + d_i} = 0, \tag{4.102}$$

$$-\sum_{i\in S_1} \frac{a_i}{(z_0 + d_i)^2} + \sum_{i\in S_2} \frac{a_i}{(-z_0 + d_i)^2} - \sum_{i\in S_3} \frac{a_i}{(-z_0 + d_i)^2} \geq 0. \tag{4.103}$$

Recall that $a_i>0$ and $z_0+d_i>0\,(i\in S_1)$. Then, we have

$$\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}<\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}, \tag{4.104}$$

$$\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}>\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}. \tag{4.105}$$

Recall that the hypothesis of this case is that $s_3\geq 2$ and for any $j\in S_2$, we have $a_j\geq \sum_{i\in S_3}a_i$. Then, by (4.104)-(4.105) and by Cauchy's inequality, we have

$$\begin{aligned}\sum_{i\in S_2}\frac{a_i^2}{(-z_0+d_i)^2}&\geq\sum_{i\in S_2}\frac{a_i}{(-z_0+d_i)^2}\sum_{i\in S_3}a_i>\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}\sum_{i\in S_3}a_i & (4.106)\\ &\geq\left(\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}\right)^2>\left(\sum_{i\in S_2}\frac{a_i}{-z_0+d_i}\right)^2, & (4.107)\end{aligned}$$

which is a contradiction.

Case 3. Recall that the hypothesis of this case is that there exists $k\in\{1,\ldots,s_2\}$ such that

$$\sum_{i=s_1+1}^{s_1+k}a_i\leq\min_{i\in S_3}\{a_i\}<\sum_{i\in S_3}a_i\leq a_{s_1+k+1}\leq\cdots\leq a_{s_1+s_2}, \tag{4.108}$$

where $a_{s_1+1}\leq a_{s_1+2}\leq\cdots\leq a_{s_1+s_2}$. Similar to the proof of Case 2, we have (4.98).

Step 1. In this step, for the $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3$ in (4.98), we prove that there exists

$$\widetilde{G}(z):=\frac{A_1}{z+D_1}+\frac{A_2}{-z+D_2}+\frac{A_3}{-z+D_3}+\theta-\sum_{i\in S_3}\frac{a_i}{-z+d_i} \tag{4.109}$$

such that

$$\widetilde{G}(\widetilde{z}_1)<0,\quad \widetilde{G}(\widetilde{z}_2)>0,\quad \widetilde{G}(\widetilde{z}_3)<0, \tag{4.110}$$

where

$$A_1>0,\quad 0<A_2\leq\min_{i\in S_3}\{a_i\},\quad A_3\geq\sum_{i\in S_3}a_i, \tag{4.111}$$

$$\widetilde{z}_i+D_1>0,\quad -\widetilde{z}_i+D_2>0,\quad -\widetilde{z}_i+D_3>0\quad (i=1,2,3), \tag{4.112}$$

$$\theta\geq 0. \tag{4.113}$$

Recall that by (3.3), for any $i\in S_1$, we have $a_i>0$. Since $\widetilde{z}_i\in I$ $(i=1,2,3)$, by (4.96), we have $\widetilde{z}_i>-\min_{i\in S_1}\{d_i\}$ $(i=1,2,3)$. Then, by Lemma 4.4, there exist $A_1,D_1,\theta_1\in R$ such that

$$\sum_{i\in S_1}\frac{a_i}{\widetilde{z}_j+d_i}=\frac{A_1}{\widetilde{z}_j+D_1}+\theta_1\quad (j=1,2,3), \tag{4.114}$$

where

$$\min_{i\in S_1}\{a_i\}\le A_1\le\sum_{i\in S_1}a_i,\quad D_1\ge\min_{i\in S_1}\{d_i\},\quad \theta_1\ge 0. \tag{4.115}$$

So, we have $A_1>0$ in (4.111) and $\widetilde{z}_i+D_1>0\,(i=1,2,3)$ in (4.112). By (4.108), let $S_2^{(1)}=\{s_1+1,\ldots,s_1+k\}$ and $S_2^{(2)}=\{s_1+k+1,\ldots,s_1+s_2\}$. So, $S_2=S_2^{(1)}\cup S_2^{(2)}$ and by (4.108), we have

$$\sum_{i\in S_2^{(1)}}a_i\le\min_{i\in S_3}\{a_i\}, \tag{4.116}$$

$$\sum_{i\in S_3}a_i\le\min_{i\in S_2^{(2)}}\{a_i\}. \tag{4.117}$$

Recall that by (3.3), for any $i\in S_2$, $a_i>0$. Since $\widetilde{z}_i\in I$ $(i=1,2,3)$, by (4.97), we have $-\widetilde{z}_j>-\mathcal{R}\ge-\min_{i\in S_2^{(1)}}\{d_i\}$ $(j=1,2,3)$. Similarly, by Lemma 4.4, there exist $A_2,D_2,\theta_2\in R$ such that

$$\sum_{i\in S_2^{(1)}}\frac{a_i}{-\widetilde{z}_j+d_i}=\frac{A_2}{-\widetilde{z}_j+D_2}+\theta_2\quad (j=1,2,3), \tag{4.118}$$

where

$$\min_{i\in S_2^{(1)}}\{a_i\}\le A_2\le\sum_{i\in S_2^{(1)}}a_i,\quad D_2\ge\min_{i\in S_2^{(1)}}\{d_i\},\quad \theta_2\ge 0. \tag{4.119}$$

Then, by (4.116), we have $0<A_2\le\min_{i\in S_3}\{a_i\}$ in (4.111) and $-\widetilde{z}_i+D_2>0\,(i=1,2,3)$ in (4.112). Recall that by (3.3), for any $i\in S_2$, $a_i>0$. Since $\widetilde{z}_i\in I$ $(i=1,2,3)$, by (4.97), we have $-\widetilde{z}_j>-\mathcal{R}\ge-\min_{i\in S_2^{(2)}}\{d_i\}$ $(j=1,2,3)$. Similarly, by Lemma 4.4, there exist $A_3,D_3,\theta_3\in R$ such that

$$\sum_{i\in S_2^{(2)}}\frac{a_i}{-\widetilde{z}_j+d_i}=\frac{A_3}{-\widetilde{z}_j+D_3}+\theta_3\quad (j=1,2,3), \tag{4.120}$$

where

$$\min_{i\in S_2^{(2)}}\{a_i\}\le A_3\le\sum_{i\in S_2^{(2)}}a_i,\quad D_3\ge\min_{i\in S_2^{(2)}}\{d_i\},\quad \theta_3\ge 0. \tag{4.121}$$

Then, by (4.117), we have $A_3\ge\sum_{i\in S_3}\{a_i\}$ in (4.111) and $-\widetilde{z}_i+D_3>0(i=1,2,3)$ in (4.112). Let $\theta:=\theta_1+\theta_2+\theta_3$. Notice that in (4.115), (4.119), and (4.121), we have $\theta_i\ge 0$. So, we have (4.113). By (4.114), (4.118), and (4.120), we have $\widehat{G}(\widetilde{z}_i)=g'(\widetilde{z}_i)$ $(i=1,2,3)$. Then, by (4.98), we have (4.110).

Step 2. Below we will deduce a contradiction from (4.110) by discussing two cases.

Case 3.1. Assume that

$$\min_{i\in S_3}\{d_i\}\le D_2. \tag{4.122}$$

Step 1. In this step, we prove that there exists $z^* \in (\widetilde{z}_1, \widetilde{z}_2)$ such that

$$\frac{A_2}{-z^*+D_2}+\frac{A_3}{-z^*+D_3}-\sum_{i\in S_3}\frac{a_i}{-z^*+d_i}<0, \tag{4.123}$$

$$\frac{A_2}{(-z^*+D_2)^2}+\frac{A_3}{(-z^*+D_3)^2}-\sum_{i\in S_3}\frac{a_i}{(-z^*+d_i)^2}\geq 0. \tag{4.124}$$

Note that by (4.110), we have $\widetilde{G}(\widetilde{z}_1)<0$, and $\widetilde{G}(\widetilde{z}_2)>0$. Note that $\widetilde{G}(z)$ is a rational function. So, there exists $z^* \in (\widetilde{z}_1, \widetilde{z}_2)$ such that

$$\widetilde{G}(z^*)=0, \tag{4.125}$$

$$\frac{d\widetilde{G}}{dz}(z^*)\geq 0. \tag{4.126}$$

Since $z^* \in (\widetilde{z}_1, \widetilde{z}_2)$, by (4.112), we have $z^*+D_1>0$. Note that by (4.111), we have $A_1>0$. By (4.109) and (4.125), we have (4.123). Notice that by (4.109), we have

$$\frac{d\widetilde{G}}{dz}=-\frac{A_1}{(z+D_1)^2}+\frac{A_2}{(-z+D_2)^2}+\frac{A_3}{(-z+D_3)^2}-\sum_{i\in S_3}\frac{a_i}{(-z+d_i)^2}. \tag{4.127}$$

So, by (4.126) and (4.127), we have (4.124).

Step 2. Let $d_q:=\min_{i\in S_3}\{d_i\}$, where $q\in S_3$. Let $S_3^*:=S_3\setminus\{q\}$. Let

$$e_2^{(1)}:=\frac{-z^*+d_q}{-z^*+D_2},\quad e_2^{(2)}:=\frac{-z^*+d_q}{-z^*+D_3},\quad e_i:=\frac{-z^*+d_q}{-z^*+d_i}$$

for any $i\in S_3^*$. In this step, we prove that

$$\sum_{i\in S_3}a_i\left(\sum_{i\in S_3^*}a_ie_i^2+a_q-A_2\big(e_2^{(1)}\big)^2\right)-\left(\sum_{i\in S_3^*}a_ie_i+a_q-A_2e_2^{(1)}\right)^2<0. \tag{4.128}$$

In fact, we multiply (4.123) by $-z+d_q$, and we can get

$$A_3e_2^{(2)}<\sum_{i\in S_3^*}a_ie_i+a_q-A_2e_2^{(1)}. \tag{4.129}$$

By (4.111), we have

$$A_3\geq\sum_{i\in S_3}a_i. \tag{4.130}$$

By (4.129) and (4.130), we have

$$A_3(e_2^{(2)})^2={A_3}^2\big(e_2^{(2)}\big)^2\frac{1}{A_3}<\left(\sum_{i\in S_3^*}a_ie_i+a_q-A_2e_2^{(1)}\right)^2\frac{1}{\sum_{i\in S_3}a_i}. \tag{4.131}$$

We multiply (4.124) by $(-z+d_q)^2$ and we can get

$$A_3\big(e_2^{(2)}\big)^2 \geq \sum_{i\in S_3^*} a_i e_i^2 + a_q - A_2\big(e_2^{(1)}\big)^2. \tag{4.132}$$

Then, by (4.131) and (4.132), we have

$$\sum_{i\in S_3^*} a_i e_i^2 + a_q - A_2\big(e_2^{(1)}\big)^2 < \left(\sum_{i\in S_3^*} a_i e_i + a_q - A_2 e_2^{(1)}\right)^2 \frac{1}{\sum_{i\in S_3} a_i},$$

which is equivalent to (4.128).

Step 3. Define

$$\mathcal{F}(y) := \left(\sum_{i\in S_3^*} a_i + y\right)\left(\sum_{i\in S_3^*} a_i e_i^2 + y - A_2\big(e_2^{(1)}\big)^2\right) - \left(\sum_{i\in S_3^*} a_i e_i + y - A_2 e_2^{(1)}\right)^2.$$

Note that $\mathcal{F}(a_q)<0$ is equivalent to (4.128). In order to deduce a contradiction, the goal of this step is to prove $F(a_q)\geq 0$. Notice that by (4.111), we have $a_q\geq A_2$. So, we only need to show that $d\mathcal{F}/dy\geq 0$ for any $y\in\mathbb{R}$ and $\mathcal{F}(A_2)\geq 0$. Note that

$$\frac{d\mathcal{F}}{dy} = \sum_{i\in S_3^*} a_i(e_i-1)^2 + A_2 e_2^{(1)} + A_2 e_2^{(1)}\big(1-e_2^{(1)}\big),$$

and note that by (4.122), we have $0<e_2^{(1)}\leq 1$. Note also that by (4.111), $A_2>0$. So,

$$\frac{dF}{dy} > \sum_{i\in S_3^*} a_i(e_i-1)^2 \geq 0.$$

Below, we prove that $\mathcal{F}(A_2)\geq 0$. Define

$$\mathcal{G}(x) := \left(\sum_{i\in S_3^*} a_i + A_2\right)\left(\sum_{i\in S_3^*} a_i e_i^2 + A_2 - A_2 x^2\right) - \left(\sum_{i\in S_3^*} a_i e_i + A_2 - A_2 x\right)^2. \tag{4.133}$$

Notice that $\mathcal{G}(1/e_2^{(1)}) = \mathcal{F}(A_2)$. So, we only need to prove that $\mathcal{G}(1/e_2^{(1)})\geq 0$. Notice that $\mathcal{G}(x)$ is a quadratic function in x, and its coefficient of x^2 is negative. So the minimum of $\mathcal{G}$ over $(0,1)$ is greater than $\mathcal{G}(0)$ and $\mathcal{G}(1)$. By Cauchy's inequality, we have

$$\mathcal{G}(0) = \left(\sum_{i\in S_3^*} a_i + A_2\right)\left(\sum_{i\in S_3^*} a_i e_i^2 + A_2\right) - \left(\sum_{i\in S_3^*} a_i e_i + A_2\right)^2 \geq 0.$$

Also,

$$\mathcal{G}(1)=\left(\sum_{i\in S_3^*}a_i+A_2\right)\sum_{i\in S_3^*}a_ie_i^2-\left(\sum_{i\in S_3^*}a_ie_i\right)^2>\sum_{i\in S_3^*}a_i\sum_{i\in S_3^*}a_ie_i^2-\left(\sum_{i\in S_3^*}a_ie_i\right)^2\geq 0.$$

So, for any $x\in(0,1]$, $\mathcal{G}(x)\geq 0$. Notice that $1/e_2^{(1)}\in(0,1]$. Then, we have $\mathcal{G}(1/e_2^{(1)})\geq 0$.

Case 3.2. Assume that

$$\min_{i\in S_3}\{d_i\}>D_2. \tag{4.134}$$

Define

$$P(z):=\widetilde{G}(z)-\theta. \tag{4.135}$$

Note that (4.110) is equivalent to

$$P(\widetilde{z}_1)<-\theta,\quad P(\widetilde{z}_2)>-\theta,\quad P(\widetilde{z}_3)<-\theta, \tag{4.136}$$

where $\widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3$. Below, we deduce a contradiction from (4.136) by discussing two cases.

Case 3.2.1. We assume that for any $z\in(\widetilde{z}_1,\widetilde{z}_3)$, we have $P(z)\leq 0$. Note that $P(z)$ is a rational function. So, if (4.136) holds, then there exists $z_0\in(\widetilde{z}_1,\widetilde{z}_3)$, such that

$$\frac{dP}{dz}(z_0)=0, \tag{4.137}$$

$$\frac{d^2P}{dz^2}(z_0)\leq 0. \tag{4.138}$$

Since $z_0\in(\widetilde{z}_1,\widetilde{z}_3)$, we have $P(z_0)\leq 0$. Then, by (4.109) and (4.135), we have

$$\frac{A_1}{z_0+D_1}+\frac{A_2}{-z_0+D_2}+\frac{A_3}{-z_0+D_3}\leq\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}. \tag{4.139}$$

Note that by (4.111) and (4.112), we have $A_1/(z_0+D_1)>0$. Then, by (4.139), we have

$$\frac{A_2}{-z_0+D_2}+\frac{A_3}{-z_0+D_3}\leq\sum_{i\in S_3}\frac{a_i}{-z_0+d_i}. \tag{4.140}$$

Note that $dP/dz=d\widetilde{G}/dz$. So, by (4.127) and (4.137), we have

$$\begin{aligned}\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^2}&=-\frac{A_1}{(z_0+D_1)^2}+\frac{A_2}{(-z_0+D_2)^2}+\frac{A_3}{(-z_0+D_3)^2}\\&\leq\frac{A_2}{(-z_0+D_2)^2}+\frac{A_3}{(-z_0+D_3)^2}.\end{aligned} \tag{4.141}$$

By (4.138), we have

$$\frac{d^2P}{dz^2}(z_0)=\frac{2A_1}{(z_0+D_1)^3}+\frac{2A_2}{(-z_0+D_2)^3}+\frac{2A_3}{(-z_0+D_3)^3}-\sum_{i\in S_3}\frac{2a_i}{(-z_0+d_i)^3}\le 0.$$

So, we have

$$\frac{A_2}{(-z_0+D_2)^3}+\frac{A_3}{(-z_0+D_3)^3}\le\sum_{i\in S_3}\frac{a_i}{(-z_0+d_i)^3}. \tag{4.142}$$

Below, we show that (4.140)-(4.142) can not hold concurrently. Let

$$\gamma:=\frac{A_3}{A_2},\quad e:=\frac{-z_0+D_3}{-z_0+D_2},$$

and for any $i\in S_3$, let

$$\gamma_i:=\frac{a_i}{A_2},\quad e_i:=\frac{-z_0+d_i}{-z_0+D_2}.$$

By (4.111), we have $\gamma_i>1$ $(i\in S_3)$ and $\gamma>\sum_{i\in S_3}\gamma_i$. By (4.112) and (4.134), we have $e_i>1(i\in S_3)$ and $e>0$. Then, we divide the both sides of (4.140) by $A_2/(-z_0+D_2)$, and we get

$$1+\frac{\gamma}{e}\le\sum_{i\in S_3}\frac{\gamma_i}{e_i}. \tag{4.143}$$

Similarly, by (4.141) and (4.142), we have

$$1+\frac{\gamma}{e^2}\ge\sum_{i\in S_3}\frac{\gamma_i}{e_i^2}, \tag{4.144}$$

$$1+\frac{\gamma}{e^3}\le\sum_{i\in S_3}\frac{\gamma_i}{e_i^3}. \tag{4.145}$$

By Lemma 4.5, (4.143)-(4.145) lead to a contradiction.

Case 3.2.2. We assume that there exists $z_0\in(\widetilde{z}_1,\widetilde{z}_3)$ such that $P(z_0)>0$. Let

$$\psi(z):=(z+D_1)P(z) \tag{4.146}$$

$$=A_1+(z+D_1)\left(\frac{A_2}{-z+D_2}+\frac{A_3}{-z+D_3}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}\right). \tag{4.147}$$

Note that $P(z_0)>0$. Recall that by (4.112), we have $\theta>0$. So, by (4.136), we have $P(\widetilde{z}_1)<0$ and $P(\widetilde{z}_3)<0$. Then by (4.146), we have

$$\psi(\widetilde{z}_1)<0,\quad \psi(z_0)>0,\quad \psi(\widetilde{z}_3)<0. \tag{4.148}$$

Note that $\psi(z)$ is a rational function. So, by (4.148), there exists $z_0^* \in (\widetilde{z}_1, \widetilde{z}_3)$, such that

$$\psi(z_0^*) > 0, \tag{4.149}$$

$$\frac{d\psi}{dz}(z_0^*) = 0, \tag{4.150}$$

$$\frac{d^2\psi}{dz^2}(z_0^*) \leq 0. \tag{4.151}$$

Step 1. In this step, we prove that by (4.148) and (4.149), we have

$$\psi(z_0^*) \leq A_1 - A_2 - A_3 + \sum_{i \in S_3} a_i. \tag{4.152}$$

We will prove the conclusion by deducing a contradiction. Let

$$w := A_1 - A_2 - A_3 + \sum_{i \in S_3} a_i. \tag{4.153}$$

Assume that

$$\psi(z_0^*) > w. \tag{4.154}$$

We first prove that for any $u \in R$, $\psi(z) = u$ has at most 4 real solutions in $(-\infty, D_2)$. Notice that $S_3 = \{s_1+s_2+1,\ldots,s_1+s_2+s_3\}$. Note that by (4.1), d_i's are distinct from each other. Assume that $d_{s_1+s_2+1} < d_{s_1+s_2+2} < \cdots < d_{s_1+s_2+s_3}$. Recall that by (4.134), we have $D_2 < d_{s_1+s_2+1}$. Notice that by (4.147), for any $i \in S_3$, we have $\lim_{z \to d_i^+} \psi(z) = -\infty$ and $\lim_{z \to d_{i+1}^-} \psi(z) = +\infty$. So, for any $i \in S_3$ satisfying $D_3 \notin (d_i, d_{i+1})$, there exists $z_i \in (d_i, d_{i+1})$ such that $\psi(z_i) = u$. Note that D_3 is located in at most one of the $s_3 - 1$ intervals (d_i, d_{i+1}) $(i \in \{s_1+s_2+1,\ldots,s_1+s_2+s_3-1\})$. Hence, $\psi(z) = u$ has at least $s_3 - 2$ real solutions in $(D_2, +\infty)$. Since the numerator of $\psi(z) - u$ is a polynomial with degree $s_3 + 2$, $\psi(z) = u$ has at most $s_3 + 2$ solutions in $(-\infty, +\infty)$. Hence, $\psi(z) = u$ has at most 4 solutions in $(-\infty, D_2)$. Let

$$u := \frac{\max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\} + \min\{A_1, \psi(z_0^*)\}}{2}. \tag{4.155}$$

Below, we prove that if (4.154) holds, then $\psi(z) = u$ has at least 5 solutions in $(-\infty, D_2)$, which will be a contradiction. We first prove that

$$u > \max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}, \quad u < \min\{A_1, \psi(z_0^*)\}. \tag{4.156}$$

In fact, by (4.155), we only need to prove

$$\min\{A_1, \psi(z_0^*)\} > \max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}.$$

By (4.148) and (4.149), we have $\psi(z_0^*) > \max\{\psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}$. Then, by (4.154), we have

$$\psi(z_0^*) > \max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}. \tag{4.157}$$

Notice that by (4.111), we have $A_1 > 0$. Then by (4.148), we have $A_1 > \max\{\psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}$. By (4.153), we have

$$A_1 - w = A_2 + A_3 - \sum_{i \in S_3} a_i. \tag{4.158}$$

By (4.111), we have $A_2 + A_3 - \sum\limits_{i \in S_3} a_i > 0$. So, we have $A_1 > w$. Then, we have

$$A_1 > \max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}. \tag{4.159}$$

Hence, by (4.157) and (4.159), we have

$$\min\{A_1, \psi(z_0^*)\} > \max\{w, \psi(\widetilde{z}_1), \psi(\widetilde{z}_3)\}.$$

Then, (4.156) holds. Note that by (4.156), we have

$$u > \psi(\widetilde{z}_1), \quad u < \psi(z_0^*), \quad u > \psi(\widetilde{z}_3). \tag{4.160}$$

Note that by (4.146) and (4.153), we have $\lim_{z \to -\infty} \psi(z) = w$, $\psi(-D_1) = A_1$, and $\lim_{z \to \min\{D_2, D_3\}^-} \psi(z) = +\infty$. Then, by (4.156), we have

$$u > \lim_{z \to -\infty} \psi(z), \quad u < \psi(-D_1).$$

By (4.112), we have $-\infty < -D_1 < \widetilde{z}_1 < z_0^* < \widetilde{z}_3 < \min\{D_2, D_3\}$. Then, we have $\psi(z) = u$ has at least 5 solutions in $(-\infty, \min\{D_2, D_3\}) \subseteq (-\infty, D_2)$, which is a contradiction. So, we must have that the inequality (4.152) holds.

Step 2. Let

$$\widetilde{\psi}(z) := \frac{A_2}{-z + D_2} + \frac{A_3}{-z + D_3} - \sum_{i \in S_3} \frac{a_i}{-z + d_i}. \tag{4.161}$$

In this step, we show that the inequality (4.152) implies

$$\widetilde{\psi}(z_0^*) = \frac{A_2}{-z_0^* + D_2} + \frac{A_3}{-z_0^* + D_3} - \sum_{i \in S_3} \frac{a_i}{-z_0^* + d_i} < 0. \tag{4.162}$$

In fact, by (4.147), it is straightforward to check that

$$\psi(z) = A_1 + (z + D_1)\widetilde{\psi}(z). \tag{4.163}$$

So, by (4.152) and (4.163), we have

$$A_2 + A_3 - \sum_{i \in S_3} a_i + (z_0^* + D_1)\widetilde{\psi}(z_0^*) \leq 0. \tag{4.164}$$

Since $z_0^* \in (\widetilde{z}_1, \widetilde{z}_3)$, by (4.112), we have $z_0^* + D_1 > 0$. By (4.111), we have $A_2 + A_3 - \sum_{i \in S_3} a_i > 0$. Then, by (4.164), we have (4.162).

Step 3. In this step, we show that by (4.150), (4.151), and (4.162), we have

$$\frac{d\widetilde{\psi}}{dz}(z_0^*)=\frac{A_2}{(-z_0^*+D_2)^2}+\frac{A_3}{(-z_0^*+D_3)^2}-\sum_{i\in S_3}\frac{a_i}{(-z_0^*+d_i)^2}>0, \tag{4.165}$$

$$\frac{d^2\widetilde{\psi}}{dz^2}(z_0^*)=\frac{2A_2}{(-z_0^*+D_2)^3}+\frac{2A_3}{(-z_0^*+D_3)^3}-\sum_{i\in S_3}\frac{2a_i}{(-z_0^*+d_i)^3}<0. \tag{4.166}$$

In fact, by (4.147), (4.161) and the chain rule, we have

$$\frac{d\psi}{dz}(z)=\widetilde{\psi}(z)+(z+D_1)\frac{d\widetilde{\psi}}{dz}(z), \tag{4.167}$$

$$\frac{d^2\psi}{dz^2}(z)=\frac{d\widetilde{\psi}}{dz}(z)+(z+D_1)\frac{d^2\widetilde{\psi}}{dz^2}(z). \tag{4.168}$$

So, by (4.150) and (4.151), we have

$$\widetilde{\psi}(z_0^*)+(z_0^*+D_1)\frac{d\widetilde{\psi}}{dz}(z_0^*)=0, \tag{4.169}$$

$$\frac{d\widetilde{\psi}}{dz}(z_0^*)+(z_0^*+D_1)\frac{d^2\widetilde{\psi}}{dz^2}(z_0^*)\le 0. \tag{4.170}$$

Recall that $z_0^*+D_1>0$. So, by (4.162) and (4.169), we have (4.165), and by (4.165) and (4.170), we have (4.166).

Step 4. In this step, we prove that (4.162), (4.165), and (4.166) can not hold concurrently. Let

$$\gamma:=\frac{A_3}{A_2},\quad e:=\frac{-z_0^*+D_3}{-z_0^*+D_2},$$

and for any $i\in S_3$, let

$$\gamma_i:=\frac{a_i}{A_2},\quad e_i:=\frac{-z_0^*+d_i}{-z_0^*+D_2}.$$

By (4.111), we have $\gamma_i>1$ and $\gamma>\sum_{i\in S_3}\gamma_i$. By (4.112) and (4.134), we have $e_i>1$ and $e>0$. Then, we divide the both sides of (4.162) by $A_2/(-z_0^*+D_2)$, and we get

$$1+\frac{\gamma}{e}\le\sum_{i\in S_3}\frac{\gamma_i}{e_i}. \tag{4.171}$$

Similarly, by (4.165) and (4.166), we have

$$1+\frac{\gamma}{e^2}\ge\sum_{i\in S_3}\frac{\gamma_i}{e_i^2}, \tag{4.172}$$

$$1+\frac{\gamma}{e^3}\le\sum_{i\in S_3}\frac{\gamma_i}{e_i^3}. \tag{4.173}$$

By Lemma 4.5, (4.171)-(4.173) lead to a contradiction. □

4.2.4 Proof of Theorem 3.1(b)(4)

According to the hypothesis of Theorem 3.1(b)(4), we assume that S_1, S_2 and S_4 are non-empty. By (4.11), we have

$$g(z) = \sum_{i \in S_1} a_i \ln(z+d_i) - \sum_{i \in S_2} a_i \ln(-z+d_i) - \sum_{i \in S_4} a_i \ln(z+d_i).$$

Define

$$\widetilde{g}(z) := -g(-z) = -\sum_{i \in S_1} a_i \ln(-z+d_i) + \sum_{i \in S_2} a_i \ln(z+d_i) + \sum_{i \in S_4} a_i \ln(-z+d_i).$$

Then, the argument is similar to the proof of Theorem 3.1(b)(2). By the proof of Theorem 3.1(b)(3) and Lemma 4.1, G admits multistability if and only if there exists a subset S_1^* of S_1, $\sum_{i \in S_4}\{a_i\} > \sum_{i \in S_1^*} a_i > \min_{i \in S_4}\{a_i\}$.

4.3 Proof of Theorem 3.1(c)

The proof is similar to (and simpler than) the proof of Theorem 3.1(b). So, we put the details in the supplementary materials¶.

4.4 Proof of Theorem 3.1(d)

In this case, only one of the four sets S_1, S_2, S_3, S_4 is non-empty. So, $g(z)$ is monotone, and hence, $g(z) = 0$ has at most one real solution. By Lemma 4.1, G admits no multistability.

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