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Failure-Informed Adaptive Sampling for PINNs, Part III: Applications to Inverse Problems

Wenbin Liu[1], Liang Yan[2,4,*], Tao Zhou[3] and Yuancheng Zhou[3,4]

[1] *Research Center for Mathematics, Beijing Normal University, Zhuhai 519087, China.*
[2] *School of Mathematics, Southeast University, Nanjing 210096, China.*
[3] *LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, AMSS, Chinese Academy of Sciences, Beijing 100190, China.*
[4] *Institute of Mathematics, Henan Academy of Sciences, Zhengzhou 450046, China.*

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Abstract. In this paper, we present a novel adaptive sampling strategy for enhancing the performance of physics-informed neural networks (PINNs) in addressing inverse problems with low regularity and high dimensionality. The framework is based on failure-informed PINNs, which was recently developed in [Gao *et al.*, SIAM J. Sci. Comput., 45(4), 2023]. Specifically, we employ a truncated Gaussian mixture model to estimate the failure probability; this model additionally serves as an error indicator in our adaptive strategy. New samples for further computation are also produced using the truncated Gaussian mixture model. To describe the new framework, we consider two important classes of inverse problems: the inverse conductivity problem in electrical impedance tomography and the inverse source problem in a parabolic system. The effectiveness of our method is demonstrated through a series of numerical examples.

AMS subject classifications: 65M32, 65N21

Key words: Inverse problem, FI-PINNs, Gaussian mixture model.

1 Introduction

Inverse problems (IPs) are ubiquitous in science and engineering and have wide applications in many research areas, such as geophysics [17], signal processing and imaging [2], computer vision [15], remote sensing and control [23], statistics [11], and machine learning [9]. These inverse problems have received increasing interest from applied mathematicians, statisticians, and engineers over the past few decades. Many inverse problems

*Corresponding author. *Email addresses:* `wbliu@uic.edu.cn` (W. Liu), `yanliang@seu.edu.cn` (L. Yan), `tzhou@lsec.cc.ac.cn` (T. Zhou), `yczhou@lsec.cc.ac.cn` (Y. Zhou)

are described by differential equations. In this work, we focus on IPs with the following form:

$$\begin{cases} \mathcal{A}\big(x;u(x),\gamma(x)\big)=0 & \text{in } \Omega, \\ \mathcal{B}\big(x;u(x),\gamma(x)\big)=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ denotes the partial differential operator defined in the domain $\Omega\subset\mathbb{R}^d$, $\mathcal{B}$ is the boundary operator on the boundary $\partial\Omega$, u represents the solution while γ represents the function that needs to be recovered.

Let y denote indirect measurements which satisfy the following formula:

$$y(x)=\mathcal{G}\big(x;u(x),\gamma(x)\big), \quad x\in\mathcal{D}_{\text{indirect}}. \tag{1.2}$$

Here $\mathcal{G}$ is the expression defining the relationship between the indirect measurements y and solution u, coefficient γ. Then solving an IP is to find a pair (u,γ) which satisfies (1.1) and (1.2).

Most traditional methods to solve IPs in (1.1) and (1.2) require the use of mesh-based techniques, such as the finite difference method (FDM) and the finite element method (FEM), to solve the involved differential equations. These methods suffer from the so-called "curse of dimensionality". Their computational cost increase exponentially with the dimensionality of the problem. In recent years, many data-driven methods [1, 14, 19] are developed to solve inverse problem. There are many advantages to use data-driven method to solve inverse problem. Firstly, data-driven methods are mesh-free, providing a significant advantage in overcoming the curse of dimensionality. Secondly, data-driven methods only require minimal code modification when applied to solving inverse problems related to their corresponding forward problems. Finally, data-driven methods provide a natural formulation for problems involving partial differential equations (PDEs) and available data. This inherent compatibility enhances their applicability to solve inverse problem.

Physics-informed neural networks [14, 19] is one of the most popular and powerful data-driven methods for solving inverse problems via the use of deep neural networks (DNNs). In PINNs, the PDEs and available data in the IP are embedded into the loss function of the neural networks. Then automatic differentiation is used to solve the IP by minimizing the loss function at a random set of points in the domain of interest. One of the most important issues in PINNs is how to select training points. In the original PINNs, uniform sampling is used and it works well for some simple problems. In order to improve the accuracy and efficiency, nonuniform sampling is taken into consideration in training points selection. Numerous adaptive non-uniform sampling strategies have been introduced in [6, 7, 14, 20], as also discussed in a recent comprehensive review [22]. One of the commonly used adaptive nonuniform sampling methods is residual-based adaptive refinement (RAR) [14]. It adds new training points in locations with large residuals. However, RAR focuses mainly on the location where the residual is large and disregards the location of smaller residual [22]. Moreover, RAR requires a large set of candidate points in order to be accurate, which makes it not effective for high dimensional