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Global Solvability and Decay Properties for a p-Laplacian Diffusive Keller-Segel Model

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Abstract. In this paper, we consider the global well-posedness of solutions to a parabolic-parabolic Keller-Segel model with p-Laplace diffusion. We first establish a critical exponent $p^*=3N/(N+1)$ and prove that when $p>p^*$, the solution exists globally for arbitrary large initial value. When $1<p\le p^*$, there exists a uniformly bounded global strong solution for small initial value, and the solution decays to zero as $t\to\infty$. This paper improves and expands the results of [Cong and Liu, Kinet. Relat. Models, 9(4), 2016], in which the parabolic-elliptic case is studied.

AMS subject classifications: 35K65, 35K92, 92C17

Key words: Keller-Segel model, p-Laplacian, strong solution, boundedness, decay rate.

1 Introduction

Chemotaxis is a biological phenomenon that describes the movement of organisms such as amoebas, bacteria, and endothelial cells, propagating rapidly in the direction of increased chemical concentration in their environment. The movement of an organism towards regions with higher (or lower) chemical concentration is referred to as positive (or negative) chemotaxis, respectively, which are defined as chemical attractants or repellents. In 1970, Keller and Segel [16] first proposed the following chemotaxis model to describe the aggregation of cellular slime molds like the Dictostelium discoideum which can be written as follows:

$$\begin{cases} u_t=\nabla\cdot\big(\phi(u)\nabla u\big)-\nabla\cdot\big(\psi(u)\nabla v\big)+f(u), \\ v_t=\Delta v-v+u, \end{cases} \tag{1.1}$$

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where u,v denote the cell density and concentration of the chemical signal secreted by cells respectively, $\phi(u)$ is the mobility of cells, $\psi(u)$ is the chemotactic sensitivity and $f(u)$ represents proliferation or death of cells. Keller-Segel model has been widely studied in the past 50 years. The main achievements of parabolic-parabolic model are described below.

By taking $\phi(u)=1,\psi(u)=u$ and $f(u)=0$, the model (1.1) is simplified to the following form:

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v), \\ v_t=\Delta v-v+u. \end{cases} \tag{1.2}$$

Osaki and Yagi [23] proved that the semilinear version of (1.2) in one-dimensional space is globally solvable for any initial value. Especially, the system in two-dimensional space has a mass critical phenomenon. That is to say, the initial mass of the cell acts as a threshold to determine the existence and explosion of solutions in a two-dimensional bounded region. Speak specifically, when the initial mass of cells u_0 satisfies the condition $\int_\Omega u(\cdot,0)dx<8\pi$, Nagai [22] asserts the model (1.2) also has the radial solution (u,v) which exists globally in time and is globally bounded. While the initial mass of cells u_0 satisfies $\int_\Omega u(\cdot,0)dx>8\pi$, Herrero and Velázquez [10] found the radially symmetric solution of model (1.2) blows up in finite time. Under the assumption that the solution does not have any symmetry, the threshold value is $\int_\Omega u(\cdot,0)dx=4\pi$. Nagai [22] showed that non-negative radially symmetric solution exists globally and is uniformly bounded for $\int_\Omega u(\cdot,0)dx<4\pi$. While Horstmann and Wang [11] further proved that the solution of (1.2) blows up in finite time or infinite time under some conditions for $\int_\Omega u(\cdot,0)dx>4\pi$. Moreover, Horstmann and Winkle [12] showed that the solution of (1.2) in the multi-dimensional space blows up in finite time for arbitrarily small initial value, whereas the solution is globally solvable for some special small initial value. We refer to [37–40] or the references therein for more details.

Generally speaking, the diffusion speed of chemical substances is faster than the migration speed of cells, therefore, the parabolic-parabolic system usually neglects the time derivative v_t, and is replaced by the following parabolic-elliptic system:

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v), \\ 0=\Delta v-v+u, \end{cases} \tag{1.3}$$

for which, Nagai [19] showed that the solution of the model (1.3) in one-dimensional space never blow up. Jäger [13] showed the global existence of solutions in $2D$ space for small initial data, and also proved the existence of radial solutions that blow up at finite time. Especially, Childress and Percus [5,6] pointed out the system in two-dimensional space has a mass critical phenomenon. And in the case of radial initial functions (u_0,v_0), the threshold number is conjectured as $\int_\Omega u(\cdot,0)dx=8\pi$. Then Nagai [19] confirmed that the system has a mass critical phenomenon, that is, when the initial mass of cells satisfies $\int_\Omega u(\cdot,0)dx<8\pi$, he proved that the radial solution (u,v) exists globally in time and is globally bounded. While the initial mass of cells satisfies $\int_\Omega u(\cdot,0)dx>8\pi$ and