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A High Order Explicit Time Finite Element Method for the Acoustic Wave Equation with Discontinuous Coefficients

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Abstract. In this paper, we propose a novel high order unfitted finite element method on Cartesian meshes for solving the acoustic wave equation with discontinuous coefficients having complex interface geometry. The unfitted finite element method does not require any penalty to achieve optimal convergence. We also introduce a new explicit time discretization method for the ordinary differential equation (ODE) system resulting from the spatial discretization of the wave equation. The strong stability and optimal hp-version error estimates both in time and space are established. Numerical examples confirm our theoretical results.

AMS subject classifications: 65M12, 65M60

Key words: Explicit time discretization, strong stability, unfitted finite element, hp error estimates.

1 Introduction

The wave equation is a fundamental equation in mathematical physics describing the phenomena of wave propagation. It finds diverse applications in science and engineering, including geoscience, petroleum engineering, and telecommunication (see [33, 34]

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and the references therein). Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and $T>0$ be the length of the time interval. We consider in this paper the acoustic wave equation

$$\begin{cases} \dfrac{1}{\rho c^2}\partial_t u=\operatorname{div}\mathbf{q}+f, \quad \rho\partial_t\mathbf{q}=\nabla u & \text{in } \Omega\times(0,T), & (1.1\text{a})\\ [\![u]\!]=0, \quad [\![\mathbf{q}\cdot\mathbf{n}]\!]=0 & \text{on } \Gamma\times(0,T), & (1.1\text{b})\\ u=0 & \text{on } \partial\Omega\times(0,T), & (1.1\text{c})\\ u(\mathbf{x},0)=u_0(\mathbf{x}), \quad \mathbf{q}(\mathbf{x},0)=\mathbf{q}_0(\mathbf{x}) & \text{in } \Omega, & (1.1\text{d}) \end{cases}$$

where u is the pressure, $\mathbf{q}$ is the speed of the displacement in the medium, and f is the source. The domain Ω is assumed to be divided by a C^2-smooth interface Γ into two nonintersecting subdomains such that $\Omega=\Omega_1\cup\Gamma\cup\Omega_2$ and $\Omega_1\subset\bar{\Omega}_1\subset\Omega$. For simplicity, we assume that the density of the medium ρ and the speed of the propagation of the wave c are piecewise constants, namely,

$$\rho=\rho_1\chi_{\Omega_1}+\rho_2\chi_{\Omega_2}, \quad c=c_1\chi_{\Omega_1}+c_2\chi_{\Omega_2},$$

where for $i=1,2$, ρ_i,c_i are positive constants and χ_{Ω_i} denotes the characteristic function of Ω_i. Here $\mathbf{n}$ is the unit outer normal to Ω_1, and $[\![v]\!]|_\Gamma:=v|_{\Omega_1}-v|_{\Omega_2}$ denotes the jump of a function v across the interface Γ.

There exists a large literature on numerical methods for solving the wave equation on conforming quadrilateral/hexahedral or triangular/tetrahedral meshes for which we refer to the monograph [21] and [23, 46] for the construction of the algorithms and the finite element error analysis. Local discontinuous Galerkin (DG) methods for the wave equation are studied in [19,44]. Optimal error estimates for sufficiently smooth solutions are proved in [19] on Cartesian meshes without using the penalty and in [44] on unstructured meshes by adding appropriate penalty terms, which leads to however a dissipative method. In [24], both dissipative and non-dissipative variants of the hybridizable DG methods are proposed, where it is shown that the dissipative method has the optimal error estimate and the non-dissipative method whose numerical flux includes the time derivative of the pressure is sub-optimal.

In order to deal with an arbitrarily shaped interface where the coefficients of the partial differential equations are discontinuous, immersed or unfitted mesh methods are developed to avoid expensive work of mesh generation using body-fitted methods in, e.g. [5, 18]. For acoustic wave equations with discontinuous coefficients, a second order immersed interface method on Cartesian meshes with suitable modification of the finite difference stencil near the interface is developed in [31]. In [2], second and third order immersed DG methods are proposed which design polynomial shape functions to approximately satisfy the interface conditions. In [12] low order and [40, 41] high order cut finite elements for solving the wave equation are studied. The small cut cell problem, that is, the small intersection of the interface and the elements of the mesh can always occur, is treated by adding penalty terms of jumps of high order derivatives over interior