CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0013

Vol. **5**, No. 4, pp. 884-913
December 2024

Revisiting Parallel Splitting Augmented Lagrangian Method: Tight Convergence and Ergodic Convergence Rate

Fan Jiang[1,2,3], Bingyan Lu[1] and Zhongming Wu[4,*]

[1] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China.*
[2] *Center for Applied Mathematics of Jiangsu Province, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China.*
[3] *Jiangsu International Joint Laboratory on System Modeling and Data Analysis, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China.*
[4] *School of Management Science and Engineering, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China.*

Received 27 April 2024; Accepted 19 October 2024

Abstract. This paper revisits the convergence and convergence rate of the parallel splitting augmented Lagrangian method, which can be used to efficiently solve the separable multi-block convex minimization problem with linear constraints. To make use of the separable structure, the augmented Lagrangian method with Jacobian-based decomposition fully exploits the properties of each function in the objective, and results in easier subproblems. The subproblems of the method can be solved and updated in parallel, thereby enhancing computational efficiency and speeding up the convergence. We further study the parallel splitting augmented Lagrangian method with a modified correction step, which shows improved performance with larger step sizes in the correction step. By introducing a refined correction step size with a tight bound for the constant step size, we establish the global convergence of the iterates and $\mathcal{O}(1/N)$ convergence rate in both the ergodic and non-ergodic senses for the new algorithm, where N denotes the iteration numbers. Moreover, we demonstrate the applicability and promising efficiency of the method with tight step size through some applications in image processing.

AMS subject classifications: 65K05, 65K10, 90C25

Key words: Convex programming, augmented Lagrangian method, optimal step size, convergence rate.

*Corresponding author. *Email addresses:* `15905154902@163.com` (F. Jiang), `lubingyan2023@163.com` (B. Lu), `wuzm@nuist.edu.cn` (Z. Wu)

1 Introduction

We consider the following separable convex minimization model with linear constraints:

$$\begin{aligned} \min \ & \sum_{i=1}^{m} f_i(x_i) \\ \text{s.t.} \ & \sum_{i=1}^{m} A_i x_i = b, \end{aligned} \tag{1.1}$$

where $f_i:\mathcal{R}^{n_i}\to(-\infty,\infty]$ $(i=1,\ldots,m)$ are proper closed convex functions, $A_i\in\mathcal{R}^{l\times n_i}$ $(i=1,\ldots,m)$ are given matrices, $b\in\mathcal{R}^l$ is a given vector, and $\sum_{i=1}^m n_i = n$. Throughout the paper, the solution set of (1.1) is assumed to be nonempty. We denote $\mathbf{x}=(x_1,x_2\ldots,x_m)$ for notational simplicity.

The augmented Lagrangian method, as presented in [23, 33], serves as a benchmark for addressing (1.1) from both theoretical and algorithmic perspectives. Let the Lagrangian function of (1.1) be

$$\mathcal{L}(x_1,\ldots,x_m,\lambda)=\sum_{i=1}^m f_i(x_i)-\lambda^\top\left(\sum_{i=1}^m A_i x_i - b\right),$$

and the augmented Lagrangian function of (1.1) be

$$\mathcal{L}_\Gamma(x_1,\ldots,x_m,\lambda)=\mathcal{L}(x_1,\ldots,x_m,\lambda)+\frac{1}{2}\left\|\sum_{i=1}^m A_i x_i - b\right\|_\Gamma^2,$$

where $\lambda\in\mathcal{R}^l$ is the Lagrange multiplier and $\Gamma\in\mathcal{R}^{l\times l}$ is a symmetric, positive definite matrix. Note that when Γ is chosen as βI, with I representing the identity matrix, the aforementioned augmented Lagrangian function simplifies to the traditional form commonly utilized in the literature. This, in turn, gives rise to the following iterative scheme for solving (1.1):

$$\begin{cases} \left(x_1^{k+1},\ldots,x_m^{k+1}\right)=\arg\min\left\{\mathcal{L}_\Gamma\left(x_1,\ldots,x_m,\lambda^k\right) \mid x_i\in\mathcal{R}^{n_i}, i=1,\ldots,m\right\}, \\ \lambda^{k+1}=\lambda^k-\Gamma\left(\sum_{i=1}^m A_i x_i^{k+1}-b\right). \end{cases} \tag{1.2}$$

The above scheme is a precise implementation of augmented Lagrangian method (ALM) for addressing problem (1.1), and thus the sequence generated by (1.2) possesses the known global convergence of the ALM. In [34], it has demonstrated that the ALM is indeed an application of the proximal point algorithm (PPA), introduced in [32], applied to the dual of (1.1) with $m=1$. However, when dealing with the separable scenario with $m\geq 2$, the execution of (1.2) may encounter challenges as the variables x_i are coupled due to the quadratic term $\|\sum_{i=1}^m A_i x_i - b\|_\Gamma^2/2$. Hence, a direct implementation of ALM is not