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Global Dynamics of a Threshold Control Discrete Population Model with Allee Effects

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Abstract. This paper investigates a threshold control discrete population model with Allee effects, characterized by density-dependent growth functions separated at a critical population threshold. The model captures diverse ecological scenarios through simple switching mechanisms while maintaining biological realism. We overcome analytical challenges in piecewise systems by developing a complete classification of five distinct dynamics, revealing how critical transitions emerge at specific parameter boundaries. One key theoretical contribution identifies the precise conditions generating persistent oscillations, a counterintuitive result demonstrating how discontinuous switching can sustain periodic behavior despite monotonic growth functions. These findings provide actionable conservation strategies, including extinction prevention protocols and sustainable harvesting policies. The framework offers both theoretical advances in piecewise dynamical systems and practical tools for ecological management, with potential applications in species conservation and ecosystem restoration.

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1 Introduction

Difference equations are remarkably universal, providing fundamental tools for modeling discrete-time dynamics in fields ranging from population ecology and economic forecasting to digital signal processing and control systems engineering [6,7,12,15,19,26–30].

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The first-order difference equation

$$x_{n+1}=f(x_n),\quad n\in\mathbb{Z}(0):=\{0,1,2,\ldots\} \tag{1.1}$$

defines a recurrence relation and represents a discrete-time dynamical system. Eq. (1.1) describes how a system evolves in steps, where x_n is the state at time n, and the function f determines the next state x_{n+1} from x_n. Despite its deceptively simple form, the first-order equation (1.1) can exhibit rich dynamical behavior, exemplified by $f(x)=rx(1-x)$, a classic discrete Logistic model

$$x_{n+1}=rx_n(1-x_n),\quad x_n\in[0,1] \tag{1.2}$$

for single species population dynamics, where x_n is the normalized population of generation n and r is the intrinsic growth rate. Eq. (1.2) is a cornerstone of nonlinear dynamics and chaos theory, whose dynamics range from stable fixed points and period-doubling cascade to chaotic behavior as the growth rate r increases, illustrating how simple nonlinear iteration rules can produce complex behavior [14].

The dynamical behavior described by Eq. (1.1) is entirely determined by the properties of the right-hand function f. Extensive research has explored the special case where f maps an interval I into itself, i.e. $f:I\to I$, as seen in [3,16] and subsequent works. However, real-world applications require more natural modeling approaches where the system's dynamics depend explicitly on the state variable x_n. This motivates us to investigate the threshold control system

$$x_{n+1}=\begin{cases} f_l(x_n), & x_n\le B,\\ f_r(x_n), & x_n>B, \end{cases} \tag{1.3}$$

which incorporates threshold control through the threshold level $B\in(0,1)$. In model (1.3), both f_l and f_r maps $[0,1]$ to $[0,1]$, which are continuous functions. This framework is particularly relevant for ecological management scenarios where different intervention strategies must be implemented based on population density levels. For example, most endangered species exhibit Allee effects, where populations below a critical threshold face increased extinction risk due to reproductive failure. In such situations, threshold control triggers interventions, such as captive breeding, before populations reach irreversible decline. Threshold control is also a fundamental strategy for achieving sustainable fisheries management by scientifically defining critical population levels B that trigger adaptive interventions.

Owing to its theoretical significance and practical implications, state-dependent difference equations have become a prominent research direction in nonlinear dynamical systems and applied mathematics, demonstrating significant progress in both theoretical analysis and practical applications in recent years. See [11, 17, 18, 20–24] and references therein. For example, Xiang *et al.* [20–23] proposed state-dependent switching discrete host-parasitoid models with an economic threshold (ET) to investigate integrated pest