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DEEP NEURAL NETWORK FOR SOLVING DIFFERENTIAL EQUATIONS MOTIVATED BY LEGENDRE-GALERKIN APPROXIMATION

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Abstract. In this paper, we propose the Legendre-Galerkin Network (LGNet), a novel machine learning-based numerical solver for parametric partial differential equations (PDEs) using spectral methods. Spectral methods leverage orthogonal function expansions, such as Fourier series and Legendre polynomials, to achieve highly accurate solutions with a reduced number of grid points. Our framework combines the advantages of spectral methods, including accuracy, efficiency, and generalization, with the capabilities of deep neural networks. By integrating deep neural networks into the spectral framework, our approach reduces computational costs that enable real-time predictions. The mathematical foundation of the LGNet solver is robust and reliable, incorporating a well-developed loss function derived from the weak formulation. This ensures precise approximation of solutions while maintaining consistency with boundary conditions. The proposed LGNet solver offers a compelling solution that harnesses the strengths of both spectral methods and deep neural networks, providing an effective tool for solving parametric PDEs.

Key words. Deep learning, neural network, spectral element method, Legendre-Galerkin method, data driven numerical method.

1. Introduction

Partial differential equations (PDEs) serve as fundamental tools for understanding natural phenomena. These mathematical equations describe diverse phenomena, ranging from fluid dynamics and electromagnetic fields to quantum mechanics and population dynamics. However, despite their significance, obtaining exact solutions for PDEs is often a formidable task. The complexity of PDEs necessitates the use of numerical methods and approximations. These techniques enable us to approximate solutions by discretizing the PDEs into algebraic systems of equations. However, traditional numerical methods often suffer from extensive computational costs, especially when generating a large number of numerical solutions as a database. In recent years, deep learning has emerged as a promising avenue for addressing the challenges associated with solving PDEs. By harnessing the power of artificial neural networks, deep learning techniques offer an alternative approach to tackle the computational inefficiencies encountered in traditional numerical methods. Through the integration of advanced machine learning algorithms and large-scale computational architectures, deep learning enables us to develop efficient numerical solvers for PDEs. In this study, we present a novel deep neural network (DNN) approach for solving differential equations using the Legendre-Galerkin approximation. Our methodology leverages accurate solutions in a supervised learning setting to find solutions, given a forcing function input, f. We employ the residual of the

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weak formulation of the differential equations (DE) as a loss function. To demonstrate the effectiveness and versatility of our approach, we apply it to various types of equations, including fluid and wave models. By evaluating the numerical performance of the DNNs, we demonstrate their ability to provide accurate and efficient solutions.

Deep learning is a class of machine learning algorithms that employs multiple layers to progressively extract higher-level features from the dataset. These networks are referred to as *deep* due to their numerous hidden layers [2, 3, 11]. Neural networks (NNs) have demonstrated remarkable effectiveness in approximating continuous functions and achieving state-of-the-art performance in various fields [4, 9, 13, 17, 11, 38, 49]. Recent studies in computer vision have explored mathematical and numerical approaches of deep neural networks to enhance adversarial robustness [20, 42]. Our research is loosely connected to these investigations as we aim to gain a deeper understanding of implementing deep neural networks for numerical solutions. The central objective of our study is to examine the capability of a deep neural network in accurately approximating and predicting numerical solutions.

Previous works have demonstrated the success of neural architectures when applied to solutions of differential equations [4, 19, 30, 1, 49, 44, 46, 33, 36, 7]. There are some popular neural-network-based methods for solving high dimensional partial differential equations such as a Deep Ritz Method (DRM) [44, 48, 34] and the Deep Galerkin Method (DRM) [36]. The authors in [4, 49] introduce data-driven discretization, a method for learning optimized approximations to PDEs based on traditional finite-volume (or difference) schemes. The algorithm uses neural networks to estimate spatial derivatives, which are optimized end to end to best satisfy the equations on a low-resolution grid. Recently, the Physics Informed Neural Networks (PINN) introduces a novel methodology for finding solutions to complex dynamical systems utilizing automatic differentiation; see e.g. [30, 28, 19, 29, 26] among many other references. In [19, 47], the authors utilize the variational form, *i.e.*, the weak form to enhance the accuracy of their network. However, PINNs are designed to predict a single instance for a given set of PDEs. Consequently, when the input instance changes, the NNs need to be retrained in order to adapt to the new input data. More recently, an alternative approach is introduced learning the solution operator of a family of PDEs, which is defined by the map from the inputCinitial conditions and boundary conditions, to the output solution functions [25, 24, 43]. Our research differs from the previous work by exploring different neural network architectures to obtain an accurate solution to the parametric PDEs. We use a deep convolutional neural network (CNN) to achieve sufficiently accurate solutions by predicting coefficients of spectral approximation based on Legendre-Galerkin methods [12, 35]. This approach is partially related to the data-driven discretization in [4, 49], but their neural networks predict coefficients of numerical derivatives such as finite difference methods. In addition, they make use of standard finite volume schemes to compute numerical fluxes. Hence, the accuracy of numerical solutions generated by the finite volume method is limited even if high resolution methods are implemented.

While both our research and the DGM share the term "Galerkin", the methodology and results differ significantly. Unlike the DGM, which solely relies on taking integrals of the differential equations without multiplying test functions, our