INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 5, Pages 716–738

doi: 10.4208/ijnam2024-1029

PARAMETRIC MODEL REDUCTION WITH CONVOLUTIONAL NEURAL NETWORKS

YUMENG WANG, SHIPING ZHOU, AND YANZHI ZHANG*

Abstract. Reduced order modeling (ROM) has been widely used to solve parametric PDEs. However, most existing ROM methods rely on linear projections, which face efficiency challenges when dealing with complex nonlinear problems. In this paper, we propose a convolutional neural network-based ROM method to solve parametric PDEs. Our approach consists of two components: a convolutional autoencoder (CAE) that learns a low-dimensional representation of the solutions, and a convolutional neural network (CNN) that maps the model parameters to the latent representation. For time-dependent problems, we incorporate time t into the surrogate model by treating it as an additional parameter. To reduce computational costs, we use a decoupled training strategy to train the CAE and latent CNN separately. The advantages of our method are that it does not require training data to be sampled at uniform time steps and can predict the solution at any time t within the time domain. Extensive numerical experiments have shown that our surrogate model can accurately predict solutions for both time-independent and time-dependent problems. Comparison with traditional numerical methods further demonstrates the computational effectiveness of our surrogate solver, especially for solving nonlinear parametric PDEs.

Key words. Parametric PDEs, reduced order modeling, convolutional autoencoder, convolutional neural network, decoupled training strategy.

1. Introduction

Parametric partial differential equations (PDEs) arise in various contexts, including control and design optimization [35], risk assessment [12], uncertainty quantification [5], and data assimilation [1]. The parameters describe physical and geometric constraints of PDEs and can manifest in various ways, such as model coefficients, initial conditions, boundary conditions, and even domain geometry. Solving parametric PDEs for every point in the parameter space of interest could be extremely costly and impractical, particularly in high-dimensional cases. For instance, in real-time applications with severely limited computation time, solving the PDE for even a single set of parameters can be prohibitively costly. To reduce computational costs, a common strategy is to employ reduced-order modeling (ROM) [19, 22, 29]. In this work, we propose a convolutional neural network (CNN) based ROM to solve parametric PDEs.

Let $\Omega \subset \mathbb{R}^d$ (for $d \geq 1$) denote an open bounded domain. We consider a general parametric PDE of the form:

$$\partial_t u(\mathbf{x}, t; \boldsymbol{\mu}) = \mathcal{L}u(\mathbf{x}, t; \boldsymbol{\mu}) + \mathcal{N}(\mathbf{x}, t, u; \boldsymbol{\mu}), \qquad \text{for } \mathbf{x} \in \Omega, \tag{1}$$

$$\mathcal{B}u(\mathbf{x}, t; \boldsymbol{\mu}) = g(\mathbf{x}, t), \qquad \text{for } \mathbf{x} \in \partial\Omega, \quad t \in (0, T], \tag{2}$$

where the solution u depends on space $\mathbf{x}$, time t, and parameters $\boldsymbol{\mu}$. Here, $\mathcal{L}$ denotes a linear differential operator, and $\mathcal{N}$ includes nonlinear terms of u, and $\mathcal{B}$ represents a linear boundary operator. The parameter $\boldsymbol{\mu} \in \Gamma$ could be, for example, the diffusion rate, Reynolds number, or boundary controller. We assume

Received by the editors on June 6, 2024 and, accepted on August 10, 2024.
2000 *Mathematics Subject Classification.* 65K10, 68T07, 93A15.
*Corresponding author.

that the parameter space $\Gamma \subset \mathbb{R}^p$ is compact, and the parametric map $\boldsymbol{\mu} \to u(\boldsymbol{\mu})$ is continuous [9, 14]. The general form in (1) covers both time-dependent and time-independent parametric PDEs. If a time-dependent problem is considered, the initial condition could also depend on parameters, i.e.

$$u(\mathbf{x}, 0; \boldsymbol{\mu}) = \phi(\mathbf{x}; \boldsymbol{\mu}), \qquad \text{for } \mathbf{x} \in \bar{\Omega}. \tag{3}$$

In this work, we mainly focus on problems where the parameters appear in the PDEs, initial conditions, and/or boundary conditions. We are interested in solving the PDE for a range or ensemble of parameters. Computing the PDE solution can be time-consuming, especially for nonlinear and time-dependent problems. Hence, using full order models (FOMs) to approximate the parametric map is impractical due to the constraints of computational time and resources. Consequently, ROMs have become popular for solving parameterized PDEs; see [2, 3, 6, 8, 22, 29] and references therein.

The main idea of ROMs is to find a low-dimensional representation of the original problem, such that it can be efficiently solved. In the literature, most reduced order modeling methods are based on linear projection (e.g., proper orthogonal decomposition (POD)-Galerkin method) [2, 3, 6–8, 11]. These methods are proved to be effective in solving problems that can be well approximated by a low-rank approximation. However, they encounter challenges in solving complex nonlinear problems, such as Kolmogorov n-width problems. In these cases, the projection-based methods become ineffective; see more discussion in [14, 23]. Various approaches have been proposed to address these challenges, such as combining traditional reduced basis methods with Bayesian nonlinear regression approach [32], and using a locally weighted proper orthogonal decomposition method [27]. Although these techniques enhance the performance of the original ROM methods to some extent, they also introduce substantial computational costs and complexity.

Recently, there has been an emerging trend of nonlinear, data-driven ROM approaches using neural networks [16, 18, 20, 23–25]. Neural network-based ROMs have been applied to study problems, such as cardiac electrophysiology [16], fluid dynamics [15], and water waves [23]. Instead of linear projection, these approaches compress high-dimensional data into a lower-dimensional latent space by using autoencoders. There are various types of autoencoders, including feedforward autoencoders [4, 10], recurrent autoencoders [13], graph autoencoders [28], and the widely used convolutional autoencoders [20, 23, 25]. The choice of autoencoder neural network structure typically depends on the specific tasks. The encoded representation in the latent space could be viewed as an approximation to the full-order models. For time-dependent problems, Long Short-Term Memory (LSTM) networks are typically used to learn the dynamics in the latent space [18, 23, 25]. It usually requires the input time series data sampled at uniform time steps.

In this work, we propose a CNN-based surrogate model for solving parametric PDEs. Our model consists of two components: a CAE and a latent-CNN; see the illustration in Figure 1. In the offline (training) phase, the encoder and decoder are trained to learn the low-dimensional representation in the latent space, while the latent-CNN maps parameters $\boldsymbol{\mu}$ and time t to the encoded solution in this latent space. We use a decoupled training strategy – the CAE and latent-CNN are trained separately – to enhance training efficiency. Moreover, we use a CNN structure in the latent space. Unlike LSTM, the CNN does not require uniform time steps in the training data, making it more flexible for handling multiscale time problems. It can also predict the solution at any time within the studied time frame. In the