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HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article introduces the hybridizable discontinuous Galerkin (HDG) approach to numerically approximate the solution of a linear hyperbolic integro-differential equation. A priori error estimates for semi-discrete and fully discrete schemes are developed. It is shown that the optimal order of convergence is achieved for both scalar and flux variables. To achieve that, an intermediate projection is introduced for the semi-discrete error analysis, and it also shows that this projection achieves convergence of order $h^{k+3/2}$ for $k \geq 1$. Next, superconvergence is achieved for the scalar variable using element-by-element post-processing. For the fully discrete error analysis, the central difference scheme and the mid-point rule approximate the derivative and the integral term, respectively. Hence, the second order of convergence is achieved in the temporal direction. Finally, numerical experiments have been performed to validate the theory developed in this article.

Key words. Hyperbolic integro-differential equation, hybridizable discontinuous Galerkin method, Ritz-Volterra projection, a priori error bounds, post-processing.

1. Introduction

Throughout this paper, we have discussed HDG method for the following model problem:

(1a)
$$u_{tt}(x,t) - \nabla \cdot \left(a(x)\nabla u(x,t) + \int_0^t b(x,t,s)\nabla u(x,s)ds \right) = f(x,t) \quad \text{in } \Omega \times (0,T],$$

(1b) $$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T],$$

(1c) $$u(x,0) = u_0(x) \ \forall \ x \in \Omega,$$

(1d) $$u_t(x,0) = u_1(x) \ \forall \ x \in \Omega,$$

where $u_{tt}(x,t) = \dfrac{\partial^2 u}{\partial t^2}(x,t)$ and $u : \Omega \times (0,T] \to \mathbb{R}$. The functions $f : \Omega \times (0,T] \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ and $u_1 : \Omega \to \mathbb{R}$ are known. We have assumed the following properties to be true on the domain Ω, it is convex, polygonal and bounded in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The known functions $a : \Omega \to \mathbb{R}$ and $b : \Omega \times (0,T] \times (0,T] \to \mathbb{R}$ satisfy the following properties: function a is positive and bounded. There exists $\alpha_0 > 0$, $M > 0$ such that $0 < \alpha_0 \leq a \leq M$, whereas, b is smooth and twice differentiable with bounded derivatives and $|b| \leq M$. Such classes of problems and nonlinear versions, thereof arise naturally in many applications, such as in viscoelasticity, see [28] and references therein.

In the literature, Pani *et al.* [2] have analyzed fully discrete schemes for time-dependent partial integro-differential equations, using energy methods, paying attention to the storage required during time-stepping. Further, errors are estimated in L^2 and H^1 norms. In [13], Saedpanah has formulated a continuous space-time finite element method of degree one for an integro-differential equation of hyperbolic

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type with mixed boundary conditions. Further, a posteriori error estimates are also established. Then, in [14], a first-order continuous space-time finite element method for a hyperbolic integro-differential equation has been formulated. Moreover, the theory is illustrated with the help of an example. In [12], Karaa *et al.* have applied DG method to (1). A priori error estimates are derived for both scalar as well as for vector variables, and the optimal rate of convergence is derived for the scalar variable and suboptimal rate of convergence for vector variables. In [3], Karaa *et al.* have discussed mixed finite element methods for the model problem (1). They have derived $L^\infty(L^2)$ and $L^\infty(L^\infty)$ error estimates and have shown to achieve optimal and quasi-optimal order of convergence, respectively, with minimal smoothness on the initial data. Later, in [15], Merad *et al.* proposed a Galerkin method based on least squares for a two-dimensional hyperbolic integro-differential equation with purely integral conditions. They have also discussed the existence and uniqueness of the solution of the model problem under specific conditions. In [4], Chen *et al.* have proposed a two-grid finite element scheme for a semi-linear hyperbolic integro-differential equation, which uses two grids to deal with the semi-linearity of the problem and achieves the same order of accuracy as that of the ordinary finite element method. Recently, Tan *et al.* [16] have applied a fully discrete two-grid finite element method on a hyperbolic integro-differential equation and achieved optimal order of convergence. The scheme has reduced the computational cost while maintaining numerical accuracy.

HDG method is a numerical technique for solving partial differential equations (PDEs) that combines the accuracy of the discontinuous Galerkin (DG) method with the computational efficiency of other finite element methods. HDG method was first introduced by Cockburn [6, 5, 7, 8], and has since been applied to a wide range of problems. In HDG method, the solution is approximated using piecewise polynomial functions, similar to the DG method, but with additional degrees of freedom that are defined at the element interfaces. These additional degrees of freedom are used to enforce the continuity of the solution across the element interfaces, which leads to a more efficient and accurate method than the standard DG method. HDG method has several advantages over other finite element methods, including the ability to handle complex geometries and nonlinear equations and the ability to achieve high-order accuracy with fewer degrees of freedom. HDG method has been successfully applied to a variety of PDE, including the Navier-Stokes equations [17, 18], the Maxwell equations [19, 20], and the advection-diffusion equation [21, 22], etc. In addition, the method has been extended to include time-dependent problems, such as the heat equation [11, 23], the wave equation [24, 25] and parabolic integro-differential equation [26]. In [11], Chabaud *et al.* have extended the analysis of HDG method and applied to second-order elliptic equations for the heat equation. They have shown that the superconvergence results hold for the heat equation when the HDG method is used to semi-discretize the equation. Further, in [23], Moon *et al.* have analyzed the method for the heat equation with nonlinear coefficients, which satisfy the Lipschitz condition. As far as the wave equations are considered, in [24], Cockburn *et al.* have analyzed the error estimates of the acoustic equation and have achieved optimal order of convergence for velocity as well as its gradient. They have also discussed the superconvergence result for the same. Stanglmeier *et al.* [25] has developed an explicit HDG method for acoustic wave equation that yields optimal convergence rates for the approximation of all the unknown variables and discussed some superconvergence properties. Recently, Jain *et al.* [26] have analyzed the HDG method for linear parabolic integro-differential