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ENERGY-CONSERVATIVE FINITE DIFFERENCE METHOD FOR THE COUPLED NONLINEAR KLEIN-GORDON EQUATION IN THE NONRELATIVISTIC LIMIT REGIME

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Abstract. In this paper, we propose an energy-conservative finite difference time domain (FDTD) method for solving the coupled nonlinear Klein-Gordon equations (CNKGEs) in the nonrelativistic limit regime, involving a small parameter $0 < \varepsilon \ll 1$ which is inversely proportional to the speed of light. Employing cut-off technique, we analyze rigorously error estimates for the numerical method. Numerical results are reported to confirm the energy-conservative property and the error results in l^2 norm and H^1 norm under different values of ε.

Key words. Coupled nonlinear Klein-Gordon equations, finite difference time domain method, energy-conservative, cut-off technique, nonrelativistic regime.

1. Introduction

The Klein-Gordon equations (KGEs), proposed in 1927 by physicists Oskar Klein and Walter Gordon, can be used to describe the motion of a spin-0 particle with the mass m. It is a fundamental equation in relativistic quantum mechanics and quantum field theory and can be regarded as the relativistic version of the Schrödinger equation.

The following coupled nonlinear Klein-Gordon equations in d-dimensions ($d = 1, 2, 3$) can be used to describe the interaction of two fields (ϕ and ψ) with the mass m_1 and the mass m_2, respectively,

$$\frac{1}{c^2}\partial_{tt}\phi - \triangle\phi + \frac{m_1^2c^2}{\hbar^2}\phi + \eta_1|\phi|^2\phi + \eta_2|\psi|^2\phi = 0, \tag{1a}$$

$$\frac{1}{c^2}\partial_{tt}\psi - \triangle\psi + \frac{m_2^2c^2}{\hbar^2}\psi + \eta_2|\phi|^2\psi + \eta_3|\psi|^2\psi = 0, \tag{1b}$$

where t is time, $\mathbf{x}$ is the spatial coordinate, $\phi := \phi(\mathbf{x}, t)$ and $\psi := \psi(\mathbf{x}, t)$ are functions representing electron-positron fields, $m_2 = \alpha m_1, 0 < \alpha \leq 1$, $\hbar$ is Planck's constant, c is speed of light, and η_1, η_2, η_3 are the interaction constants of electron-positron fields. Introduce the notions

$$\tilde{t} := \frac{t}{t_s},\ \tilde{\mathbf{x}} := \frac{\mathbf{x}}{\mathbf{x}_s},\ u(\tilde{\mathbf{x}}, \tilde{t}) := \frac{\phi(\mathbf{x}, t)}{\phi_s}, v(\tilde{\mathbf{x}}, \tilde{t}) := \frac{\psi(\mathbf{x}, t)}{\psi_s}, \tag{2}$$

plugging (2) into (1a) - (1b), we have

$$\frac{\mathbf{x}_s^2}{c^2t_s^2}\partial_{tt}u - \triangle u + \frac{m_1^2c^2\mathbf{x}_s^2}{\hbar^2}u + \eta_1\phi_s^2\mathbf{x}_s^2|u|^2u + \eta_2^2\psi_s^2\mathbf{x}_s^2|v|^2u = 0, \tag{3a}$$

$$\frac{\mathbf{x}_s^2}{c^2t_s^2}\partial_{tt}v - \triangle v + \frac{\alpha m_1^2c^2\mathbf{x}_s^2}{\hbar^2}v + \eta_2\phi_s^2\mathbf{x}_s^2|u|^2v + \eta_3\psi_s^2\mathbf{x}_s^2|v|^2v = 0, \tag{3b}$$

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where $t_s, \mathbf{x}_s, \phi_s, \psi_s$ are the dimensionless time unit, length unit, and filed unit. Setting $\varepsilon = \frac{\hbar}{cm_1\mathbf{x}_s} = \frac{\mathbf{x}_s}{ct_s} = \frac{v_s}{c}$, $\beta_1 = \eta_1\phi_s^2\mathbf{x}_s^2$, $\beta_2 = \eta_2^2\psi_s^2\mathbf{x}_s^2 = \eta_2^2\phi_s^2\mathbf{x}_s^2$, $\beta_3 = \eta_3\psi_s^2\mathbf{x}_s^2$, $\phi_s = \psi_s = \frac{1}{\mathbf{x}_s\sqrt{\eta}}$ with $\eta = \min\{\eta_1, \eta_2, \eta_3\}$, the dimensionless CNKGEs are obtained

$$\varepsilon^2\partial_{tt}u - \triangle u + \frac{1}{\varepsilon^2}u + \beta_1|u|^2u + \beta_2|v|^2u = 0, \tag{4a}$$

$$\varepsilon^2\partial_{tt}v - \triangle v + \frac{\alpha}{\varepsilon^2}v + \beta_2|u|^2v + \beta_3|v|^2v = 0, \tag{4b}$$

with initial conditions

$$u(\mathbf{x}, 0) = \xi(\mathbf{x}), \qquad \partial_t u(\mathbf{x}, 0) = \frac{1}{\varepsilon^2}\zeta(\mathbf{x}), \tag{4c}$$

$$v(\mathbf{x}, 0) = \rho(\mathbf{x}), \qquad \partial_t v(\mathbf{x}, 0) = \frac{\alpha}{\varepsilon^2}\eta(\mathbf{x}). \tag{4d}$$

Here $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ are unknown wave functions with temporal wavelength of $\mathcal{O}(\varepsilon^2)$, where $\varepsilon \in (0, 1)$ is a constant inversely proportional to the light-speed constant c. In addition, $\xi(\mathbf{x})$, $\rho(\mathbf{x})$, $\zeta(\mathbf{x})$ and $\eta(\mathbf{x})$ are given functions that are independent of ε. Fundamentally, nonlinear wave phenomena occur in various areas of physical science. The KGEs, widely applied in quantum and particle physics, have garnered significant attention in researching solitons and condensed matter physics [12], the interaction of solitons in plasma collisions [13], the recurrence of initial states [18], and lattice nonlinear dynamics [14]. Schiff [32] made efforts to consider nuclear saturation and shell structure in terms of many-body forces which were derived from mesons obeying a nonlinear wave equation.

The system (4) is time symmetric or time reversible. Additionally, under periodic or homogeneous Dirichlet boundary conditions, the system (4) is energy-conservative in the sense that

$$E(t) \equiv E(0), \qquad t > 0, \tag{5}$$

where $E(t)$ is the total energy defined by

$$\begin{aligned} E(t) = \int_\Omega \Big[&\varepsilon^2(\partial_t u)^2 + (\nabla u)^2 + \frac{1}{\varepsilon^2}u^2 + \varepsilon^2(\partial_t v)^2 + (\nabla v)^2 \\ &+ \frac{\alpha}{\varepsilon^2}v^2 + \frac{\beta_1}{2}u^4 + \beta_2 u^2v^2 + \frac{\beta_3}{2}v^4\Big]\, d\mathbf{x}. \end{aligned} \tag{6}$$

It is well-known that conservative numerical schemes consistently outperform nonconservative ones. The crux of their superiority lies in their ability to preserve important invariant properties, allowing for a more detailed representation of physical processes. From this perspective, the numerical simulation can be measured by the extent to maintain the invariant properties of the original continuous model. To achieve an appropriate numerical method that ensures energy conservation, the classical approach involves constructing a fully implicit scheme, which always bring significant challenges for convergence analysis. In this paper, we analyze rigourously the unconditional convergence results for the energy-conservative implicit scheme.

In the regime of $O(1)$-speed of light, where the parameter $\varepsilon > 0$ is fixed, the KGEs have garnered substantial interest, experiencing a notable surge in both analytical and numerical research. Along the analytical front, Scott [30] outlined several physical implementations and described the construction of analog models. The global classical solutions of the KGEs were investigated in [20, 33]. Moreover, the Cauchy problem for the KGEs were studied, we refer the readers to [19, 31] and therein references. Researchers proposed and analyzed standard finite difference