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FOURIER CONVERGENCE ANALYSIS FOR FOKKER-PLANCK EQUATION OF TEMPERED FRACTIONAL LANGEVIN-BROWNIAN MOTION AND NONLINEAR TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. Fourier analysis works well for the finite difference schemes of the linear partial differential equations. However, the presence of nonlinear terms leads to the fact that the method cannot be applied directly to deal with nonlinear problems. In the current work, we introduce an effective approach to enable Fourier methods to effectively deal with nonlinear problems and elaborate on it in detail by rigorously proving that the difference scheme for two-dimensional nonlinear problem considered in this paper is strictly unconditionally stable and convergent. Further, some numerical experiments are performed to confirm the rates of convergence and the robustness of the numerical scheme.

Key words. Time-fractional Fokker-Planck model, $L1$ scheme, Nonlinearity, Fourier stability-convergence analysis.

1. Introduction

Anomalous dynamics are ubiquitous in the nature world, especially in the complex system, the applications of which have a broad range, including physics [2], chemistry [14], and biology [32], etc. Unlike the classic mathematical and physical model for describing the diffusion, the anomalous diffusion processes no longer obey Fourier's or Fick's law [20, 27, 28]. We usually distinguish between normal and anomalous diffusive processes according to the mean squared displacement (MSD) (see [34] and the references therein), the MSD of the anomalous diffusing species $\langle x^2(t)\rangle$ scales as the following nonlinear power law, i.e.,

$$\langle x^2(t)\rangle \sim \kappa_\beta t^\beta,$$

where β is the anomalous diffusion index and κ_β the diffusion coefficient. According to β the anomalous diffusions are distinguished into subdiffusion if $0 < \beta < 1$, normal diffusion if $\beta = 1$, and superdiffusion if $\beta > 1$. Especially, we call it underballistic hyperdiffusion, ballistic diffusion, and hyperballistic diffusion as $1 < \beta < 2$, $\beta = 2$, and $\beta > 2$, respectively; see, e.g., [21]. Several effective methods are restored to describe the anomalous subdiffusive transport processes, including continuous time random walk model, fractal diffusion equation, fractional Klein-Kramers equation, and fractional Brownian and Langevin motion [6, 8, 9, 16, 22], etc.

In this paper, we aim to give an efficient proof idea to extend the Fourier method to deal with the nonlinear problems with nonlinear term $f(u)$. For this purpose, we consider the following two-dimensional nonlinear time fractional Fokker-Planck

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equation

(1)

$$\begin{aligned}{}_0^C D_t^\beta u(x,y,t) =& \frac{\overline{A}}{\beta}\Big[\, t\ {}_0D_t^{1-\beta} \triangle\ u(x,y,t)\Big] - (A,A)\cdot \nabla u(x,y,t) + f(u(x,y,t),x,y,t),\\ & 0 \le x,\ y \le L,\ 0 \le t \le T,\end{aligned}$$

where $0 < \beta < 1$, $\overline{A}$ and A are positive constants, and (A, A) is a two-dimensional vector. The time fractional Caputo derivative operator ${}_0^C D_t^\beta$ is defined by [24]

$$ {}_0^C D_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\frac{\partial u(s)}{\partial s}ds,$$

and the time fractional Riemann-Liouville derivative operator ${}_0D_t^{1-\beta}$ is defined as [24]

$$ {}_0D_t^{1-\beta} u(t) = \frac{1}{\Gamma(\beta)}\frac{\partial}{\partial t}\int_0^t (t-s)^{\beta-1}u(s)ds,$$

where $\Gamma(z) := \int_0^\infty s^{z-1}e^{-s}ds$ (for $\Re(z) > 0$) denotes the Gamma function. Here, the solution $u(x, y, t)$ of model (1) represents the probability density function of particle position (see, e.g., [6, 31]).

For the problem (1), the theoretical analysis will be challenged as the right-hand side of the equation contains the term $(t\ {}_0D_t^{1-\beta}\triangle u)$. Thus, in this paper, we first consider the case of $f(u) = -\kappa u + h(x, y, t)$ $(\kappa > 0)$ in model (1), i.e.,

$$\begin{aligned}{}_0^C D_t^\beta u(x,y,t) =& \frac{\overline{A}}{\beta}\Big[\, t\ {}_0D_t^{1-\beta} \triangle\ u(x,y,t)\Big] - (A,A)\cdot \nabla u(x,y,t)\\ & - \kappa u(x,y,t) + h(x,y,t),\ 0 \le x,\ y \le L,\ 0 \le t \le T,\end{aligned} \tag{2}$$

with the boundary conditions

$$\begin{aligned} u(0,y,t) = \varphi_1(y,t),\ u(L,y,t) = \varphi_2(y,t),\ 0 \le\ y \le\ L,\ 0 \le\ t \le T,\\ u(x,0,t) = \psi_1(x,t),\ u(x,L,t) = \psi_2(x,t),\ 0 \le\ x \le\ L,\ 0 \le\ t \le T,\end{aligned} \tag{3}$$

and the initial condition

$$u(x,y,0) = \phi(x,y),\quad 0 \le\ x,y \le\ L. \tag{4}$$

Then in Section 4, we consider the general two-dimensional nonlinear time fractional sub-diffusion problem

$${}_0^C D_t^\beta u = p\triangle u + (q,q)\cdot\nabla u + f(u,x,y,t), \tag{5}$$

where $p > 0$, $q \in R$.

Due to their wide applications, fractional partial differential equations (FPDEs) have generated much interest in developing stable and accurate numerical methods as well as rigorous mathematical and numerical analysis; see [12, 15] and the references therein. We know that although some analytical solutions of FPDEs are expressed in terms of some special functions, these special functions are always difficult to evaluate numerically. This has naturally led to the rapid development of various effective numerical methods, including finite difference methods [5], finite element methods [7, 13], finite volume methods [11], spectral method [10], and collocation methods [17], etc. Among the existing approaches, the finite difference approximation to the fractional derivative seems to be the most studied one. And the $L1$ method [18] and Grünwald Letnikov formula [29] are effective discretization methods and are widely used in discretization of fractional differential operators.

As we all know, the Fourier stability analysis works well and is popular for the finite difference schemes of the linear partial differential equations with constant