INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 4, Pages 556–584

doi: 10.4208/ijnam2025-1024

STABILITY AND SUPERCONVERGENCE OF A SPECIAL Q_1-FINITE VOLUME ELEMENT SCHEME OVER QUADRILATERAL MESHES

YANHUI ZHOU

Abstract. This paper studied the stability and superconvergence of a special isoparametric bilinear finite volume element scheme for anisotropic diffusion problems over quadrilateral meshes, where the scheme is obtained by employing the edge midpoint rule to approximate the line integrals in classical Q_1-finite volume element method. It can be checked that the scheme is identical to the standard five-point difference scheme for a special case. By element analysis approach, we suggest a sufficient condition to guarantee the stability of the scheme. This condition has an analytic expression, which covers the traditional $h^{1+\gamma}$-parallelogram and some trapezoidal meshes with any full anisotropic diffusion tensor. Moreover, based on the h^2-uniform quadrilateral mesh assumption, we proved the superconvergence $|u_I - u_h|_1 = \mathcal{O}(h^2)$, where u_I is the isoparametric bilinear interpolation of exact solution u, and u_h is the numerical solution. As a by product, we obtained the optimal H^1 and L^2 error estimates. Finally, the theoretical results are verified by some numerical experiments.

Key words. Q_1-finite volume element scheme, stability, superconvergence, H^1 and L^2 error estimates, anisotropic diffusion equation.

1. Introduction

The finite volume element (FVE) method (FVEM) is also called as generalized difference method [1] or box method [2]. Since FVEM possesses local conservation law and other advantages, it has been attracted many researchers attention, see the book [3] and review papers [4, 5]. For the Poisson equation, the element stiffness matrix of linear FVEM is identical to the corresponding linear finite element method (FEM) on arbitrary triangular meshes, and the coercivity result follows [6, 7]. Under the coercivity result, the optimal H^1 error analysis can be established by a standard technique. Further, [8, 9] proved the optimal L^2 error estimate on general triangular meshes, and [10, 11] applied linear FVEM to some more complicated problems. More relevant studies of high order FVEMs are presented in [12, 13, 14, 15, 16] and so on.

However, the development of standard isoparametric bilinear FVEM (Q_1-FVEM) over quadrilateral meshes is still lags far behind. To ensure the coercivity result, most work require quasi-parallelogram mesh condition [17, 18, 19]. In 2020, [20] suggested a sufficient condition to guarantee the coercivity, and this condition covers the traditional quasi-parallelogram mesh, but regrettably it not cover arbitrary trapezoidal mesh. Once the coercivity result is established, then the optimal H^1 error analysis of classical Q_1-FVEM is trival. On the other hand, by approximating the line integrals in classical Q_1-FVEM at geometric center of the quadrilateral, [21] constructed a symmetric Q_1-FVE scheme, such that the global stiffness matrix is symmetric. Recently, [22], [23] and [24] employed trapezoidal, midpoint and Simpson rules to approximate the line integrals respectively, and the coercivity results of these new schemes are validated over traditional quasi-parallelogram and some

Received by the editors on July 17, 2024 and, accepted on January 24, 2025.
2000 *Mathematics Subject Classification.* 35J25, 65N12, 65N15.

trapezoidal meshes. Further, based on Wachspress generalized barycentric coordinate, in 2023 [25] designed a polygonal FVEM to solve the anisotropic diffusion problems and presented an optimal H^1 error estimate. For more studies of FVEM over quadrilateral meshes, we refer the readers to [26, 27, 28, 29] for incomplete references.

Under the coercivity result and H^1 error estimate, we may investigate the L^2 error and superconvergence of FVEM solution. By employing the barycenters of triangles to construct the dual mesh, [30] shown that the difference between linear FEM solution and FVEM solution over triangular mesh is of second order in energy norm. On the other hand, by using Taylor's expansion, [21] presented the optimal L^2 error estimate and superconvergence of a special symmetric Q_1-FVE scheme over uniform rectangular meshes. Assume that the quadrilateral mesh is h^2-uniform, and by using the geometric centers of quadrilaterals to construct the dual mesh, [31] proved that the difference between classical Q_1-FVEM solution and the interpolation of exact solution is also of second order in energy norm. More studies of the error analysis can be found in [32, 33, 34, 35] and so on.

In this paper, we employ the value at edge midpoint to approximate the line integrals in classical Q_1-FVEM for solving anisotropic diffusion problems on general convex quadrilateral meshes. By element analysis approach, we suggest a sufficient condition to ensure the stability. This sufficient condition has an analytic expression, which only involves the anisotropic diffusion coefficient and the geometry of mesh. This leads to that for any full diffusion tensor and convex quadrilateral mesh, we can directly judge whether this sufficient condition is satisfied. More interesting is that, this condition covers the traditional $h^{1+\gamma}$-parallelogram and some trapezoidal meshes with full anisotropic diffusion coefficient.

To study the superconvergence of the error $u_I - u_h$ in energy norm, we decompose it into two parts by using the coercivity result. The first part is the error $u_I - u$, then under h^2-uniform quadrilateral mesh assumption and the superconvergence of bilinear interpolation on two adjacent quadrilateral elements, this error can be analyzed by some analysis techniques. Moreover, by Taylor's expansion, we can analyze the second part $u - u_h$. Thanks to these findings, we obtain the second order superconvergence result. As a result, we get that u_h converges to u with optimal convergence rates 1 and 2 under H^1 and L^2 norms, and the superconvergence results of u_h at geometric centers, interior vertices and edge midpoints which are all second order in an average gradient norm. We mention that [17] has pointed out that, if we use the edge midpoint rule to approximate the line integrals, then in some special cases, the new scheme is identical to the standard five-point difference scheme. However, its theoretical analysis has not been established. Thus, the novelty of this paper is that, under some mesh assumptions, we presented the stability and superconvergence for the special scheme.

The rest of this paper is organized as follows. In Section 2, we present a special isoparametric bilinear finite volume element scheme over general convex quadrilateral meshes. The stability and superconvergence results of the constructed scheme are shown in Sections 3 and 4 respectively. Some numerical experiments are reported in Section 5 to verify the theoretical results, and the conclusion is given in Section 6. In Section 8 (Appendix A), we give some lemmas that available in the literature. Some discussions for the assumption (**A1**) are presented in Section 9 (Appendix B).

Throughout the paper, C will be denote a generic constant that could change from one occurrence to the other. To avoid repetition, we sometimes write "$A \lesssim B$"