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A DIFFUSE DOMAIN APPROXIMATION WITH TRANSMISSION-TYPE BOUNDARY CONDITIONS II: GAMMA–CONVERGENCE

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Abstract. Diffuse domain methods (DDMs) have gained significant attention for solving partial differential equations (PDEs) on complex geometries. These methods approximate the domain by replacing sharp boundaries with a diffuse layer of thickness ε, which scales with the minimum grid size. This reformulation extends the problem to a regular domain, incorporating boundary conditions via singular source terms. In this work, we analyze the convergence of a DDM approximation problem with transmission-type Neumann boundary conditions. We prove that the energy functional of the diffuse domain problem Γ–converges to the energy functional of the original problem as $\varepsilon \to 0$. Additionally, we show that the solution of the diffuse domain problem strongly converges in $H^1(\Omega)$, up to a subsequence, to the solution of the original problem, as $\varepsilon \to 0$.

Key words. Partial differential equations, phase-field approximation, diffuse domain method, diffuse interface approximation, transmission boundary conditions, gamma-convergence, reaction-diffusion equation.

1. Introduction

This work is a follow-up to [16] in which we applied formal asymptotics to analyze the approximation of solutions of partial differential equations (PDEs) posed in a domain with complex geometries using a diffuse domain approach. This paper focuses on the rigorous variational analysis of the approximation process, where in addition to model approximation, we prove convergence of corresponding solutions.

PDEs posed in domains with complex geometries arise in various applications, including materials science, fluid dynamics, and biology. In many practical problems, these domains may have intricate boundaries, evolving interfaces, or irregular shapes that complicate numerical discretization and analysis. Traditional numerical approaches often require conformal meshes that accurately capture domain boundaries. Constructing such meshes can be computationally expensive and challenging, especially in scenarios where the domain evolves over time or has small-scale geometric features.

To circumvent these difficulties, diffuse domain methods (DDMs) have emerged as versatile approaches. These methods (i) embed the original complex domain into a larger, simpler computational domain, like a square or a cube, and (ii) introduce a phase field function to smoothly approximate the characteristic function of the original domain. The governing PDEs are then modified with additional penalization terms that enforce consistency between the diffuse domain approximation and the original sharp-interface formulation. By avoiding the need for complex meshing and allowing for efficient numerical implementation, DDMs have become a widely used technique in various applications, such as phase-field modeling, where they support

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simulations of complex phenomena in fields like biology (e.g., [9, 14, 13, 11, 17, 3]), fluid dynamics (e.g., [4, 20, 5, 2, 8, 19]), and materials science (e.g., [21, 17, 18, 10]).

In [16] we have studied the asymptotic convergence of the diffuse domain approximation problem in one-dimensional space. In addition, we have provided numerical simulations and discussed their outcomes in relation to our analytical result. In this paper, we prove the Γ-convergence of the energy functional associated with the diffuse domain approximation and the convergence of corresponding solutions in the strong $H^1(\Omega)$–topology, in any dimension. For motivation and background on diffuse domain problems, as well as asymptotic convergence analysis and numerical experiments, we refer the readers to [16] and the references therein.

To be precise, we study the following two-sided boundary value problem in an open cuboidal domain Ω: Find a function $u_0 : \Omega \to \mathbb{R}$ defined as

$$u_0(x) = \begin{cases} u_1(x), & \text{if } x \in \Omega_1 \subset \Omega, \\ u_2(x), & \text{if } x \in \Omega_2 = \Omega \setminus \overline{\Omega_1}, \end{cases}$$

where $u_1 : \Omega_1 \to \mathbb{R}$ and $u_2 : \Omega_2 \to \mathbb{R}$ satisfy

$$\begin{aligned} -\Delta u_1 + \gamma u_1 &= q, \quad \text{in } \Omega_1, \\ -\alpha \Delta u_2 + \beta u_2 &= h, \quad \text{in } \Omega_2, \\ u_1 &= u_2, \quad \text{on } \partial\Omega_1, \\ -\nabla(u_1 - \alpha u_2) \cdot \boldsymbol{n}_1 &= \kappa u_1 + g, \quad \text{on } \partial\Omega_1, \\ \alpha \nabla u_2 \cdot \boldsymbol{n}_2 &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, we assume the following:

(1) Ω_1 is a bounded open subset of $\mathbb{R}^n$ with a compact C^3–boundary $\partial\Omega_1$ satisfying $\overline{\Omega_1} \subset \Omega$ and $\partial\Omega_1 \cap \partial\Omega = \varnothing$, and $\Omega_2 := \Omega \setminus \overline{\Omega_1}$ (see Figure 1);
(2) $\boldsymbol{n}_1$ denotes the outward-pointing unit normal vector on $\partial\Omega_1$, and $\boldsymbol{n}_2$ denotes the outward-pointing unit normal vector on $\partial\Omega$.
(3) $h, q \in L^2(\Omega)$ and $g \in H^1(\Omega)$ are given functions;
(4) α, β, γ are given positive constants, and κ is a given nonnegative constant.

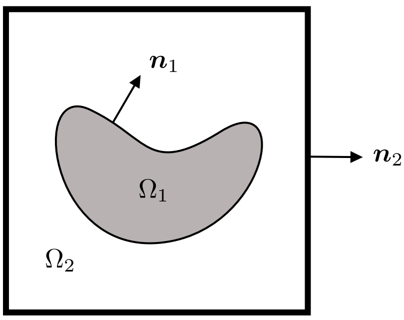

FIGURE 1. A domain Ω_1 is covered by a larger cuboidal domain Ω, with $\Omega_2 := \Omega \setminus \overline{\Omega}_1$.

The boundary conditions across the interface $\partial\Omega_1$ are called transmission-type boundary conditions, ensuring continuity of function values across the interface while allowing a jump in flux due to underlying physical mechanisms. A solution u_0 of the two-sided problem (1) corresponds to a minimizer of the associated energy functional $\mathcal{E}_0$, defined by

$$\mathcal{E}_0[u] = \int_\Omega \left[\frac{1}{2}(D_0|\nabla u|^2 + c_0 u^2) - f_0 u\right] dx + \int_{\partial\Omega_1} \left(\frac{1}{2}\kappa u^2 + gu\right) dS, \tag{2}$$