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AUTO-STABILIZED WEAK GALERKIN FINITE ELEMENT METHODS FOR BIHARMONIC EQUATIONS ON POLYTOPAL MESHES WITHOUT CONVEXITY ASSUMPTIONS

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Abstract. This paper introduces an auto-stabilized weak Galerkin (WG) finite element method for biharmonic equations with built-in stabilizers. Unlike existing stabilizer-free WG methods limited to convex elements in finite element partitions, our approach accommodates both convex and non-convex polytopal meshes, offering enhanced versatility. It employs bubble functions without the restrictive conditions required by existing stabilizer-free WG methods, thereby simplifying implementation and broadening application to various partial differential equations (PDEs). Additionally, our method supports flexible polynomial degrees in discretization and is applicable in any dimension, unlike existing stabilizer-free WG methods that are confined to specific polynomial degree combinations and 2D or 3D settings. We demonstrate optimal order error estimates for WG approximations in both a discrete H^2 norm for $k \geq 2$ and an L^2 norm for $k > 2$, as well as a sub-optimal error estimate in L^2 when $k = 2$, where $k \geq 2$ denotes the degree of polynomials in the approximation.

Key words. Weak Galerkin, finite element methods, auto-stabilized, non-convex, polytopal meshes, bubble function, weak Laplacian, biharmonic equation.

1. Introduction

In this paper, we propose an auto-stabilized weak Galerkin finite element method with built-in stabilizers suitable for non-convex polytopal meshes, specifically applied to biharmonic equations with Dirichlet and Neumann boundary conditions. Specifically, we seek to determine an unknown function u such that

$$
\begin{aligned}
\Delta^2 u =& f, && \text{in} \quad \Omega,\\
u =& \xi, && \text{on} \quad \partial\Omega,\\
\frac{\partial u}{\partial \mathbf{n}} =& \nu, && \text{on} \quad \partial\Omega,
\end{aligned} \tag{1}
$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Note that the domain Ω considered in this paper can be of any dimension d.

The variational formulation of the model problem (1) can be formulated as follows: Find an unknown function $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = \xi$ and $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \nu$, and the following equation

$$
(\Delta u, \Delta v) = (f, v), \qquad \forall v \in H_0^2(\Omega), \tag{2}
$$

where $H_0^2(\Omega) = \{v \in H^2(\Omega) : v|_{\partial\Omega} = 0, \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = 0\}$.

The weak Galerkin finite element method marks a significant advancement in numerical solutions for PDEs. This innovative approach redefines or approximates differential operators within a framework akin to the distribution theory tailored for

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piecewise polynomials. Unlike conventional techniques, the WG method alleviates the usual regularity constraints on approximating functions by employing carefully crafted stabilizers. Extensive research has demonstrated the WG method's versatility across various model PDEs, bolstered by a substantial list of references [9, 10, 36, 40, 11, 12, 13, 14, 38, 42, 4, 35, 18, 8, 22, 44, 30, 34, 31, 32, 33, 37, 39, 6, 20, 47, 46, 41, 7], underscoring its potential as a powerful tool in scientific computing. What sets WG methods apart from other finite element approaches is their use of weak derivatives and weak continuities to create numerical schemes based on the weak formulations of the underlying PDE problems. This structural versatility makes WG methods exceptionally effective across a wide range of PDEs, ensuring both stability and precision in their approximations.

A significant innovation within the weak Galerkin methodology is the "Primal-Dual Weak Galerkin (PDWG)" approach. This novel method addresses difficulties that traditional numerical strategies often encounter [15, 16, 1, 2, 3, 17, 23, 24, 43, 5, 26, 27, 25, 28, 29]. PDWG interprets numerical solutions as constrained optimization problems, with the constraints mimicking the weak formulation of PDEs through the application of weak derivatives. This innovative formulation leads to the derivation of an Euler-Lagrange equation that integrates both the primary variables and the dual variables (Lagrange multipliers), thereby creating a symmetric numerical scheme.

This paper introduces a straightforward formulation of the weak Galerkin finite element method for biharmonic equations that operates on both convex and non-convex polytopal meshes without the use of stabilizers. The key trade-off for eliminating stabilizers involves using higher-degree polynomials for computing the discrete weak Laplacian operator, which may impact practical applicability. Unlike existing stabilizer-free WG schemes limited to convex elements [45], our method accommodates non-convex polytopal meshes, preserving the size and global sparsity of the stiffness matrix while significantly reducing programming complexity. Theoretical analysis confirms optimal error estimates for WG approximations in both the discrete H^2 norm for $k \geq 2$ and the L^2 norm for $k > 2$, along with a sub-optimal error estimate in L^2 when $k = 2$, where $k \geq 2$ is the polynomial degree in the approximation.

Our method introduces several significant enhancements over the stabilizer-free weak Galerkin finite element method for biharmonic equations presented by [45]. The key contributions are summarized as follows: **1. Theoretical Foundation for Non-Convex Polytopal Meshes:** Our method provides a theoretical foundation for an auto-stabilized WG scheme that handles convex and non-convex elements in finite element partitions through the innovative use of bubble functions, while the existing stabilizer-free WG method [45] is limited to convex meshes. This enhances the practical applicability of our method, making it more versatile for real-world computational scenarios. **2. Superior Flexibility with Bubble Functions:** Unlike the method in [45], which is limited by restrictive conditions imposed in the analysis, our approach employs bubble functions as a critical analysis tool without these constraints. This flexibility allows our method to generalize to various types of PDEs without the complexities imposed by such conditions, thereby simplifying the implementation process. **3. Dimensional Versatility:** Our method is applicable in any dimension d, whereas the method in [45] is confined to 2D or 3D settings. This broader applicability makes our method suitable for higher-dimensional problems. **4. Adaptable Polynomial Degrees:** Our method