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A UNIFIED ANALYSIS FRAMEWORK FOR UNIFORM STABILITY OF DISCRETIZED VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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Abstract. We provide a unified analysis framework for discretized Volterra integrodifferential equations by considering the ϑ-type convolution quadrature, where different ϑ corresponds to different schemes. We first derive the long-time l^∞ stability of discrete solutions, and then prove a discrete Wiener-Lévy theorem to support the analysis of long-time l^1 stability. The methods we adopt include the integral transforms in the Stieltjes sense, the complex analysis techniques, and a linear algebra approach for an indirect estimate of intricate terms. Meanwhile, we relax the commonly-used regularity assumption of the initial data in the literature by novel treatments. Numerical simulations are performed to substantiate the theoretical findings.

Key words. Volterra integrodifferential equation, ϑ-type convolution quadrature, uniform stability, long-time behaviour, completely monotonic kernels.

1. Introduction

1.1. Problem formulation and motivation. This work considers the unified analysis framework for the temporal discretization of the following Volterra integrodifferential equation [2,3,21,26,27]

$$\frac{\partial u}{\partial t}+\int_0^t \chi(t-r)Au(r)dr=0, \quad t>0;\ u(0)=u_0, \tag{1}$$

where A is a positive self-adjoint linear operator defined in a dense subspace $\mathrm{D}(A)$ of the real Hilbert space $\mathbf{H}$ with a complete eigensystem $\{\gamma_q,\varphi_q\}_{q=1}^\infty$; $u_0\in\mathbf{H}$ and the kernel $\chi(t)$ on $(0,\infty)$ satisfies that

$$\chi \text{ is completely monotonic}, \quad \chi\in L^1_{\mathrm{loc}}(0,\infty), \quad 0\le\chi(\infty)<\chi(0^+)\le\infty. \tag{2}$$

Problem (1) plays an important role in various fields such as the simple shearing motions or torsion of a rod in viscoelasticity and the dynamic behavior of the velocity field of a 'linear' homogeneous isotropic incompressible viscoelastic fluid [31,34], and extensive mathematical and numerical analysis can be found in [9,11,14–20,30,33,36,37,43,44,46–48]. In particular, investigating the long-time behavior of the solutions to model (1) is critical and challenging. For the continuous case, it was proved in [4,5] that the L^1 stability of the solutions to model (1) $\int_0^\infty\|u(r)\|dr\le C\|u_0\|$ holds, in which $C>0$ is independent of $u(t)$ and $\|\cdot\|$ indicates the norm in $\mathbf{H}$ defined via the inner product $(\cdot,\cdot)$ of $\mathbf{H}$. The uniform L^1 behavior of the exponential decay of the solutions to model (1) was proved in [12]. For the discretized problems, several works have considered the asymptotic l^1 analysis of (1) with a completely monotonic kernel. Xu studied the backward Euler temporal discretization for model (1) based on a first-order convolution quadrature and proved the l^1 stability [39] and the l^1 convergence [40]. A second-order temporal finite difference approach was investigated with the stability [41] and the convergence [42] proved for model

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(1). Harris and Noren [13] utilized the backward-Euler method and deduced the uniform l^1 stability for (1).

This work intends to provide a unified analysis framework for temporal discretization of problem (1) by considering the convolution quadrature method [1, 6, 7, 23] of ϑ-type, where different ϑ corresponds to different discretizations. In particular, $\vartheta = 1$ relates to the backward Euler method, while $\vartheta = 1/2$ corresponds to the Crank-Nicolson scheme. Compared with these classical methods, the ϑ-type methods not only serve as a mathematical generalization, but also unify the analysis of backward Euler and Crank-Nicolson methods. In [10, 45], the corresponding studies for their applications in discretizing integrodifferential equations are far from mature. In particular, how to analyze the long-time stability of the numerical solutions to model (1) under the ϑ-type convolution quadrature method in order to characterize the long-time behaviour remains untreated in the literature due to its complexity, which motivates the current study.

1.2. ϑ-type convolution quadrature. Define the Laplace transform by $\widehat{w}(z) = \int_0^\infty \mathrm{e}^{-zt} w(t)dt$ for $\Re(z) > \lambda$ where $w(t)\mathrm{e}^{-\lambda t} \in L^1(\mathbb{R}^+)$ for all $\lambda > 0$ and $\Re(\cdot)$ denotes the real part of the complex number. We fix the time step size $k > 0$ and $t_n = nk$ with $n \geq 0$ and generate numerical solution $U^n \approx u(t_n)$ such that for $\vartheta \in [1/2, 1]$ (see [24, 25])

$$\widehat{u}\left(\frac{\delta(\xi)}{k}\right) = k\sum_{n=0}^{\infty} U^n \xi^n, \quad \xi \in \mathbb{C}, \quad |\xi| < 1; \quad \delta(\xi) = \frac{1-\xi}{\vartheta + (1-\vartheta)\xi}. \tag{3}$$

Applying the Laplace transform to (1), we obtain $\widehat{u}(z) = \left(\widehat{\chi}(z)A + zI\right)^{-1} u_0$ for $\Re(z) \geq 0$. Combining this equation and (3) we obtain

$$k\sum_{j=0}^{\infty} w_j(k)\xi^j \sum_{n=j}^{\infty} AU^{n-j}\xi^{n-j} + \sum_{n=0}^{\infty} \frac{1-\xi}{\vartheta + (1-\vartheta)\xi} U^n \xi^n = u_0, \tag{4}$$

where the quadrature weight $w_j(k)$ is determined by [25]

$$\widehat{\chi}\left(\frac{\delta(\xi)}{k}\right) = \sum_{j=0}^{\infty} w_j(k)\xi^j, \quad \xi \in \mathbb{C}, \quad |\xi| \leq 1. \tag{5}$$

We swap the summation indices of the first left-hand side term of (4) to get

$$k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^n + \sum_{n=0}^{\infty} \frac{1-\xi}{\vartheta + (1-\vartheta)\xi} U^n \xi^n = u_0.$$

Further, we arrange the above formula to obtain

$$\begin{aligned}\vartheta k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^n + (1-\vartheta)k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^{n+1} + U^0 \\ + \sum_{n=1}^{\infty}(U^n - U^{n-1})\xi^n = [\vartheta + (1-\vartheta)\xi]u_0.\end{aligned} \tag{6}$$

By comparing the coefficients of ξ^n $(n \geq 0)$ on both sides of (6), we obtain the ϑ-type convolution quadrature scheme of model (1) for $\frac{1}{2} \leq \vartheta \leq 1$,

$$U^0 = \vartheta\left(I + \vartheta k w_0(k)A\right)^{-1} u_0, U^1 = \left(I + \vartheta k w_0(k)A\right)^{-1}\left[\vartheta^{-1}I - \vartheta k w_1(k)A\right]U^0, \tag{7}$$

$$\frac{U^n - U^{n-1}}{k} + \vartheta\sum_{j=0}^{n} w_j(k)AU^{n-j} + (1-\vartheta)\sum_{j=0}^{n-1} w_j(k)AU^{n-1-j} = 0, \quad n \geq 2. \tag{8}$$