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ANALYSIS OF A TYPE II THERMAL PROBLEM INVOLVING A VISCOELASTIC BEAM

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Abstract. In this work, we will study, from both analytical and numerical points of view, a nonlocal problem involving a thermoviscoelastic beam which has been modeled by using the type II thermal law. In the first part, we will show that this problem has a unique solution and that the solutions decay exponentially by using the theory of linear semigroups. We will also prove that the semigroup of contractions is not differentiable and the impossibility of localization, that is, we will obtain that the unique solution which can vanish in an open nonempty set is the null solution. In the second part, we will focus on the numerical approximation of a variational formulation of the thermomechanical problem. By using the finite element method and the implicit Euler scheme to approximate the spatial variable and to discretrize the time derivatives, respectively, a fully discrete scheme will be introduced. Then, we will prove a discrete stability property and we will provide an a priori error analysis. The linear convergence of the approximations will be deduced whenever the continuous solution is regular enough. Finally, some numerical results will be presented to demonstrate the numerical convergence and the exponential decay of the discrete energy.

Key words. Viscoelastic beam, type II thermoelasticity, finite elements, a priori error estimates, numerical simulations.

1. Introduction

The study of thermoelastic materials has received a large number of contributions dealing with both quantitative or qualitative aspects. In these studies, we can find results about the existence, uniqueness and stability as well as the numerical behavior of the solutions. In this sense, it is suitable to recall several contributions concerning plate thermoelastic problems [1, 2, 3, 4, 10, 12, 13, 17, 18, 20, 22]. In these references, the conservative component is mechanical and the dissipative aspect is thermal. The objective of this work is to follow these lines but it is worth noting that, here, we consider a nonlocal thermoviscoelastic bar formulated with a conservative heat conduction model.

Therefore, it is adequate to recall that the idea of non-locality in elasticity was introduced by Eringen [8, 9] and that this mechanism suggests a regularization of the solutions [11]. From the physical point of view, this is introduced to take into account the effects at long distances.On the other hand, from the thermal point of view, it is worth noting that Green and Naghdi proposed three thermoelastic theories depending on the type of heat conduction (see, for details, [14, 15, 16]). In this paper, we consider the so-called type II thermal law, which does not allow the energy dissipation, leading to a conservative behavior. That is, in this paper we change the role of the mechanical and thermal aspects and we consider the nonlocal effect.

The paper is structured as follows. In the next section we will describe the problem that we will study in this work. Then, in Sections 3 and 4 the existence

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and the energy decay of the solutions will be obtained by means of the theory of linear semigroups, although we will also prove that the semigroup is not differentiable. In Section 5, we will see that the unique solution which vanishes for every time $t \geq t_0 \geq 0$ is the null solution. Then, in Section 6 we will focus on the numerical approximation of a variational formulation of the above thermomechanical problem. The fully discrete scheme will be provided by using the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property and an a priori error analysis will be shown by using a discrete version of Gronwall's inequality. The linear convergence of the approximations will be deduced under some additional regularity of the continuous solution. Finally, in Section 7 we will present two numerical tests: the first one will demonstrate the convergence of the discrete solution when both spatial parameter and time step tend to zero, leading to the theoretical linear convergence. The second example will show the exponential decay of the discrete energy.

2. Preliminaries

In this paper, we will consider a thermoviscoelastic bar which occupies the domain $(0,\pi)$, and modeled by using the Euler-Bernoulli theory when the heat conduction is determined by the type II Green-Naghdi theory (see [16]).

Therefore, we will study the system:

$$(1) \qquad \left.\begin{array}{l} \rho u_{tt} - \tau u_{ttxx} + \mu u_{xxxx} + \mu^* u_{txxxx} - \beta \alpha_{txx} = 0, \\ c\alpha_{tt} - \kappa \alpha_{xx} + \beta u_{txx} = 0 \end{array}\right\} \quad \text{in } (0,\pi)\times(0,T).$$

In the previous equations, T is the final time, u is the mechanical displacement and α is the thermal displacement which satisfies $\alpha_t = \theta$ (the temperature). We will also consider the boundary conditions:

$$(2) \qquad \left.\begin{array}{l} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \\ \alpha(0,t) = \alpha(\pi,t) = 0 \end{array}\right\} \quad \text{for a.e. } t \in (0,T),$$

and the initial conditions:

$$(3) \qquad \left.\begin{array}{ll} u(x,0) = u_0(x), & u_t(x,0) = v_0(x) \\ \alpha(x,0) = \alpha_0(x), & \alpha_t(x,0) = \theta_0(x) \end{array}\right\} \quad \text{for a.e. } x \in (0,\pi).$$

In this work, we will assume that ρ, τ, μ, μ^*, c, κ and β are constants and, in general, that ρ, τ, μ, μ^*, c and κ are positive. Moreover, when we study the energy decay of the solutions to our problem, we will also assume that $\beta \neq 0$.

We can recall that our problem is similar to the one studied analytically in [21]. However, here we consider the inertial effects determined by the parameter τ. Furthermore, in this work we emphasize in the numerical aspect of the problem, which was not considered in [21].

3. Existence of solutions

In this section, we will show the existence and uniqueness of solutions to the problem determined by system (1), boundary conditions (2) and initial conditions (3). First, we will write our problem as a Cauchy problem in an adequate Hilbert space. Let us denote $U = (u, v, \alpha, \theta)$ and consider the Hilbert space:

$$\mathcal{H} = H_0^1(0,\pi) \cap H^2(0,\pi) \times H_0^1(0,\pi) \times H_0^1(0,\pi) \times L^2(0,\pi),$$

where $L^2(0,\pi)$, $H_0^1(0,\pi)$ and $H^2(0,\pi)$ represent the usual Sobolev spaces.

We can define the scalar product in $\mathcal{H}$ associated to the norm:

$$\|U\|_{\mathcal{H}}^2 = \mu\|u_{xx}\|^2 + \rho\|v\|^2 + \tau\|v_x\|^2 + \kappa\|\alpha_x\|^2 + c\|\theta\|^2.$$