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LOCATING NATURAL SUPERCONVERGENT POINTS OF FINITE ELEMENT METHODS IN 3D

ZHIMIN ZHANG AND RUNCHANG LIN

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Abstract. In [20], we analytically identified natural superconvergent points of function values and gradients for several popular three-dimensional polynomial finite elements via an orthogonal decomposition. This paper focuses on the detailed process for determining the superconvergent points of pentahedral and tetrahedral elements.

Key Words. Finite element methods, three-dimensional problems, natural superconvergence, pentahedral elements and tetrahedral elements.

1. Introduction

Superconvergence of the finite element (FE) method is a phenomenon that, at special *a priori* points, the convergent rate of FE approximations exceeds what is globally possible. By *natural superconvergence*, we mean that a higher order accuracy is achieved without applying any recovering or averaging techniques in the FE solution.

There have been many studies concerning with superconvergence of FE methods since the 1970s [8]. Books and survey papers have been published. For the literature, we refer to [1, 6, 7, 11, 12, 13, 17, 21] and references therein.

In [3], Babuška *et al.* predicted derivative superconvergent points for the Laplace equation, the Poisson equation, and linear elasticity equations. They reduced the problem of finding natural superconvergent points to the problem of finding intersections of certain polynomial contours. The actual superconvergent points were determined by computer programs without explicitly constructing those polynomials. Therefore, this approach is called *"computer-based"* proof.

Later, Zhang proposed an analytic approach. By an orthogonal decomposition under local rectangular and brick (hexahedral) meshes [18, 19], he constructed explicitly the polynomials for determining the superconvergent points in the "computer-based" proof and obtained analytically, superconvergence results in FE solutions for the tensor product, serendipity, and *intermediate* families. In [14], the authors studied natural superconvergence of derivatives and function values under

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four mesh patterns of triangular elements via the same approach. Our results confirmed those provided in [3] by the "computer-based" proof. Moreover, many new superconvergence results were obtained.

Recently, the authors have reported some natural superconvergence results for several 3D FE, which are used to approximate sufficiently smooth solutions of the Poisson and the Laplace equations [20]. The main theorems in the "computer-based" proof are generated to the 3D problems. Several FE meshes and spaces are studied in details. In particular, Lagrangian and serendipity elements of both hexahedron and pentahedron (triangular prism) are considered. Two patterns of tetrahedral elements are also discussed.

We notice that there are many cases involved in the investigation [20], and the process in 3D problems is rather complicated. Therefore, detailed explanation is necessary for retrieval and a better recognition of the method. Since the situation for the hexahedral elements has been thoroughly studied in [18, 20], the present paper shall focus on the approach of locating superconvergent points of pentahedral and tetrahedral elements.

In Section 2, some notations are introduced. The FE meshes and spaces are also described. The procedure of how to determine superconvergent points are illustrated through detailed examples. In particular, an example for tetrahedral element is provided in Section 3. Several points of comparison between two tetrahedral partitions are also addressed. Examples for the Lagrangian and serendipity pentahedral elements are given in Section 4 and 5, respectively. A summary for superconvergence results of the discussed elements of order 2 and 3 is given in the last section.

2. Preliminaries

2.1. Notations. Assume that a 3D FE mesh is locally translation invariant. Then, as shown in [3, 20], the task of finding superconvergent points can be narrowed down in the *master cell*, or equivalently in the *reference cell* $K = [-1, 1]^3$. In the context, (ξ, η, ζ) is used for the standard Euclidean coordinates in K.

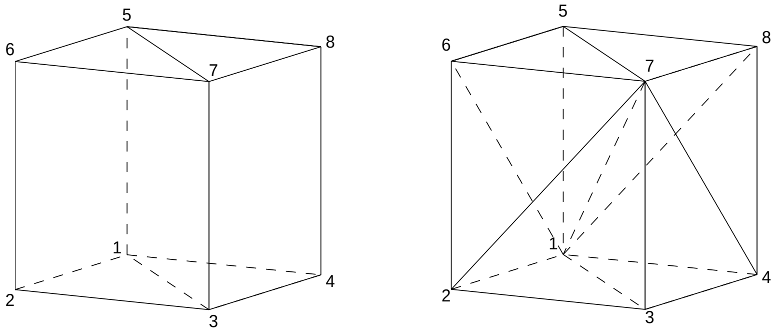
Let $V_n(K)$ and $V_n^\pi(K)$ be the *FE local space* and the *periodic FE local space* of order n defined on K, respectively. Let Π_n be the space defined by

$$\Pi_n = \text{Span}\left\{ \xi^i \eta^j \zeta^k \,|\, 0 \le i, j, k \le n,\ 0 \le i + j + k \le n \right\}.$$

Write $\Pi_n(K)$ the restriction of Π_n on K. Associated with a particular partition of K, we denote $\Pi_n^w(K)$ (resp. $\Pi_n^\pi(K)$) the set of *piecewisely continuous* (resp. *periodic piecewise continuous*) polynomials of degree not greater than n.

We need to point out that, for pentahedral elements, $\Pi_n^w(K)$ is a proper subset of $V_n(K)$, and $\Pi_n^\pi(K) \subsetneq V_n^\pi(K)$; For tetrahedral elements, $\Pi_n^w(K) = V_n(K)$, and $\Pi_n^\pi(K) = V_n^\pi(K)$.

Denote the set of $(n+1)$th degree *homogeneous harmonic polynomials* in three variables by $\mathcal{H}_{n+1}$. Then it is shown that $\dim \mathcal{H}_{n+1} = 2n + 3$ [2]. Furthermore, by the *Kelvin transform*, we can obtain an explicit basis of $\mathcal{H}_{n+1}$.



(a) Pentahedral Elements (b) Tetrahedral Elements

Figure 1. Partition of K for Pentahedral and Tetrahedral Elements

Now, for $n \geq 1$, we define $\Phi_{n+1}(K)$ the subset of $\Pi^{\pi}_{n+1}(K)$, which consists of functions ψ decomposable into $\Pi_{n+1}(K)$ and $\Pi^{w}_{n}(K)$, so that the following conditions hold:

$$\int_K \psi = 0; \qquad \int_K \nabla\psi \cdot \nabla v = 0, \ \forall v \in \Pi^{\pi}_{n}(K). \tag{2.1}$$

It is straightforward to show that $\dim \Phi_{n+1}(K) = \dim \Pi_{n+1}(K) - \dim \Pi_n(K)$. This proposition implies that, for each monomial of degree $(n+1)$, there is a unique function in $\Phi_{n+1}(K)$ corresponding to it. Similar statement holds for polynomials in $\mathcal{H}_{n+1}(K)$. We define the following set.

$$\Phi^{\mathcal{H}}_{n+1}(K) = \{\, \psi \in \Phi_{n+1}(K) \mid \psi = \chi + r, \ \chi \in \mathcal{H}_{n+1}(K), \, r \in \Pi^{w}_{n}(K) \,\} . \tag{2.2}$$

Moreover, $\dim \Phi^{\mathcal{H}}_{n+1}(K) = \dim \mathcal{H}_{n+1}(K)$.

Two sets of auxiliary functions will also be used. Let P_k be the *Legendre polynomial* of degree k on $[-1,1]$. Define

$$\begin{gathered} \phi_0(x) = 1, \quad \phi_1(x) = x, \\ \phi_k(x) = \int_{-1}^{x} P_{k-1}(t)\, \mathrm{d}t, \quad k = 2, 3, \ldots \end{gathered} \tag{2.3}$$

When $k \geq 2$, $\phi_k(x)$ are polynomials vanishing at $x = \pm 1$. Therefore, we may define polynomials $\varphi_k(x)$ so that

$$\phi_k(x) = \frac{1}{4}(1 - x^2)\varphi_{k-2}(x), \quad k = 2, 3, \ldots \tag{2.4}$$

We note that both ϕ_k and φ_k are defined for $k = 0, 1, \ldots$; In addition, the subscripts indicate the degrees of the polynomials.

2.2. Finite Element Meshes and Spaces. As described in [20], *hierarchic* bases are used for the local FE spaces $V_n(K)$. Four types of modal functions are involved: nodal shape functions, edge modes, face modes, and internal modes (see also [16]).

In pentahedral mesh, we assume that each cell (cube) is divided into two prisms. The master cell is mapped to K as in Figure 1 (a). We will refer the element $\{\xi \geq \eta\}$ as e_1 and the other element as e_2.

Lagrangian and serendipity pentahedral elements are used, whose explicit basis functions are defined in [20]. Then the bases of $V_n^\pi(K)$ can be constructed.

For tetrahedral meshes, we consider the partition scheme shown in Figure 1 (b) in this paper. It is also called *Kuhn partition* of the unit cube [4]. One way to visualize the partition is cutting K by 3 planes: $\xi = \eta$, $\eta = \zeta$, and $\zeta = \xi$. We shall note that the superconvergence results in [4, 5, 9, 10] are all for tetrahedral elements partitioned in this scheme. For convenience, we denote $e_1 = \{\xi \geq \eta \geq \zeta\}$, $e_2 = \{\xi \geq \zeta \geq \eta\}$, $e_3 = \{\zeta \geq \xi \geq \eta\}$, $e_4 = \{\zeta \geq \eta \geq \xi\}$, $e_5 = \{\eta \geq \zeta \geq \xi\}$, and $e_6 = \{\eta \geq \xi \geq \zeta\}$. These six tetrahedra are isomorphic and symmetric.

Bases of the FE local spaces are defined in [16, 20]. Then, the periodic bases of $V_n^\pi(K)$ can be obtained from $V_n(K)$ [20].

3. Tetrahedral element

For tetrahedral element, we have $\Pi_n(K) \subset \Pi_n^w(K) = V_n(K)$. Since $V_n(K)$ does not contain any polynomial of degree greater than n, it follows that

$$\Pi_{n+1}(K) \setminus V_n(K) = \Pi_{n+1}(K) \setminus \Pi_n(K),$$

whose dimension is $(n+2)(n+3)/2$. Moreover,

$$\Phi_{n+1}(K) \backslash V_n(K) = \Phi_{n+1}(K)$$

with the same dimension. According to the main theorems in [20], to study the superconvergence, we shall first determine $\Phi_{n+1}(K) \backslash V_n(K)$, which is $\Phi_{n+1}(K)$ in this case.

3.1. Determining $\Phi_{n+1}(K)$. Suppose K is partitioned into tetrahedra as in Figure 1 (b). Let u be a monomial of degree $(n+1)$ in three variables. We define an interpolation operator I_n such that:

(i) $I_n u \in V_n(K)$, and

$$I_n u(n_p) = u(n_p), \quad p = 1, \ldots, 8; \tag{3.5}$$

(ii) Along each edge l,

$$\int_l (u - I_n u) r^j \, \mathrm{d}r = 0, \quad j = 0, 1, \cdots, n-2; \tag{3.6}$$

(iii) On each face S,

$$\int_S (u - I_n u) r^j s^k \, \mathrm{d}r \, \mathrm{d}s = 0, \quad j,k \geq 0, \quad j+k = 0, 1, \cdots, n-3; \tag{3.7}$$

(iv) In each tetrahedron T,

$$\int_T (u - I_n u) \xi^j \eta^k \zeta^l \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta = 0, \quad j,k,l \geq 0, \quad j+k+l = 0, 1, \cdots, n-4, \tag{3.8}$$

where n_p stands for the pth node of K. We shall note that $u - I_n u$ is a periodic function in K [17]. Thus, a periodic FE approximation z^π of $u - I_n u$ is obtained by solving

$$\begin{gathered}\int_K \nabla(u - I_n u - z^\pi) \cdot \nabla v = 0, \quad \forall v \in V_n^\pi(K); \\ \int_K (u - I_n u - z^\pi) = 0.\end{gathered} \tag{3.9}$$

Then $u - I_n u - z^\pi$ is the unique correspondence of u in $\Phi_{n+1}(K)$. Letting u run through all monomials of degree $(n+1)$, we get a basis of $\Phi_{n+1}(K)$.

As an example, we show the detailed process for the case of $n = 2$. We need to find a basis of $\Phi_3(K)$. For instance, for monomial $\xi\eta\zeta$, the interpolation is

$$I_n(\xi\eta\zeta) = \begin{cases} -\xi\eta + \eta\zeta + \eta & \text{in } e_1 \\ -\xi\zeta + \eta\zeta + \zeta & \text{in } e_2 \\ \xi\eta - \xi\zeta + \xi & \text{in } e_3 \\ \xi\eta - \eta\zeta + \eta & \text{in } e_4 \\ \xi\zeta - \eta\zeta + \zeta & \text{in } e_5 \\ -\xi\eta + \xi\zeta + \xi & \text{in } e_6 \end{cases}. \tag{3.10}$$

Solve the FE problem (3.9). We get the associated correction term $z^\pi|_{\xi\eta\zeta} = 0$. Finally, $\xi\eta\zeta - I_n(\xi\eta\zeta) - z^\pi|_{\xi\eta\zeta}$ is the function in $\Phi_3(K)$ associated to $\xi\eta\zeta$. Similarly, we can obtain the functions in $\Phi_3(K)$ associated to all monomials of degree 3, which form a basis of $\Phi_3(K)$.

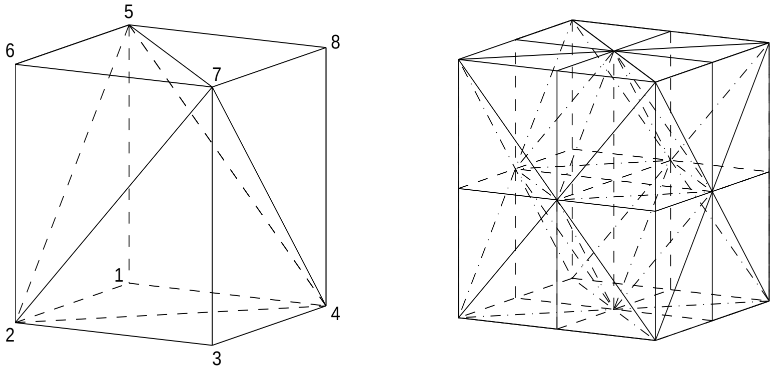
Recall that the six tetrahedra obtained from Kuhn partition are isomorphic and symmetric. Therefore, it is sufficient to determine the expressions for the piecewisely defined basis functions in only one tetrahedron, say e_1. The expressions in the other tetrahedra are obtained by symmetry. A basis of $\Phi_3(K)$ for tetrahedral finite element are (in e_1):

$$\begin{aligned} &\psi_3^1 = \xi^3 - \xi, && \psi_3^6 = \eta^2\zeta + \eta^2 - \eta\zeta - \eta, \\ &\psi_3^2 = \eta^3 - \eta, && \psi_3^7 = \eta^2\xi - \eta^2 + \xi\eta - \eta, \\ &\psi_3^3 = \zeta^3 - \zeta, && \psi_3^8 = \zeta^2\xi - \zeta^2 + \zeta\xi - \zeta, \\ &\psi_3^4 = \xi^2\eta + \xi^2 - \xi\eta - \xi, && \psi_3^9 = \zeta^2\eta - \zeta^2 + \eta\zeta - \zeta, \\ &\psi_3^5 = \xi^2\zeta + \xi^2 - \zeta\xi - \xi, && \psi_3^{10} = \xi\eta\zeta + \xi\eta - \eta\zeta - \eta. \end{aligned} \tag{3.11}$$

3.2. Superconvergence for the Poisson Equation. The main theorems in [20] indicate that, when $n > 1$, the nth order function value superconvergent points for the Poisson equation are the intersections of the basis functions of $\Phi_{n+1}(K)\backslash V_n(K)$; When $n \geq 1$, the derivative superconvergent points for the Poisson equation are the common zeros of the corresponding derivatives of the basis functions of $\Phi_{n+1}(K)\backslash V_n(K)$.

For the tetrahedral elements, $\Phi_{n+1}(K)\backslash V_n(K) = \Phi_{n+1}(K)$. We continue on the example in §3.1. For function value superconvergence in e_1, we need to find the intersections of all the polynomials in (3.11). It is straightforward to verify that, in this particular case, the vertices and the midpoint of edges of e_1 are the function value superconvergent points. In addition, by symmetry, one concludes that the *symmetry points* (namely, the vertices, the midpoint of edges, the centroid of faces, and the centroid) of K are the function value superconvergent points in K.

On the other hand, the derivative superconvergent points are the common zeros of the according derivatives of these functions. For instance, the ξ-derivative superconvergent points in e_1 are common zeros of the ξ-derivatives of the 10 functions



(a) Partition of a Cube　　　(b) Partition of K

FIGURE 2. Tetrahedral Partition Scheme 2

in (3.11); i.e. we need to solve the equation system consists of

$$\begin{aligned} \frac{\partial\psi_3^1}{\partial\xi} &= 3\xi^2 - 1 = 0, & \frac{\partial\psi_3^7}{\partial\xi} &= \eta^2 + \eta = 0, \\ \frac{\partial\psi_3^4}{\partial\xi} &= 2\xi\eta + 2\xi - \eta - 1 = 0, & \frac{\partial\psi_3^8}{\partial\xi} &= \zeta^2 + \zeta = 0, \\ \frac{\partial\psi_3^5}{\partial\xi} &= 2\xi\zeta + 2\xi - \zeta - 1 = 0, & \frac{\partial\psi_3^{10}}{\partial\xi} &= \eta\zeta + \eta = 0, \end{aligned} \tag{3.12}$$

since the ξ-derivatives of the other polynomials are 0. It turns out that the two second order Gaussian points $(\pm\sqrt{3}/3,\ -1,\ -1)$ of the edge l_{12} are the ξ-derivative superconvergent points in e_1. By symmetry, it follows that the second order Gaussian points of the edges parallel to the ξ axis are the ξ-derivative superconvergent points in K.

3.3. Superconvergence for the Laplace Equation. We have analogous theorems for the Laplace equation [20]; namely, the nth order function value (resp. derivative) superconvergent points for the Laplace equation are the common zeros (resp. of the corresponding derivative) of the basis functions of $\Phi_{n+1}^{\mathcal{H}}(K)\backslash V_n(K)$.

For the tetrahedral elements, $\Phi_{n+1}^{\mathcal{H}}(K)\backslash V_n(K) = \Phi_{n+1}^{\mathcal{H}}(K)$. Continue the example of quadratic element. We need a basis of $\Phi_3^{\mathcal{H}}(K)$. From §2.1, we know that the dimension of $\Phi_3^{\mathcal{H}}(K)$ is the same as that of $\mathcal{H}_3$, which is 7. A basis of $\Phi_3^{\mathcal{H}}(K)$ is

$$\begin{gathered} \psi_3^1 - 3\psi_3^7, \quad \psi_3^2 - 3\psi_3^4, \quad \psi_3^3 - 3\psi_3^5, \\ \psi_3^1 - 3\psi_3^8, \quad \psi_3^2 - 3\psi_3^9, \quad \psi_3^3 - 3\psi_3^6, \\ \psi_3^{10}. \end{gathered} \tag{3.13}$$

In general, the degree $(n+1)$ terms of a function in $\Phi_{n+1}^{\mathcal{H}}(K)$ must be harmonic.

Now, the superconvergent points can be obtained by solving the proper equation systems. For this particular case, the superconvergence results for the Laplace equation are the same as those for the Poisson equation.

3.4. Remarks.

Remark 3.1. Another partition scheme yielding 5 tetrahedra is also studied in [20]. See Figure 2 (a). Noting that one cube partitioned in this scheme is not translation invariant. Thus, we need 8 cubes in a patch (master cell). See Figure 2

(b). To determine superconvergence, we use the same approach as described for the Kuhn partition. However, at this time, there are 40 elements involved and the process is much more complicated.

Remark 3.2. We compare the superconvergence results for tetrahedral elements in the two partition schemes. In the Kuhn partition, superconvergence always occurs at the symmetry points, which can be predicted by the *symmetry theory* [15]. In addition, the results for the Poisson and the Laplace equations are the same. The example of quadratic element is provided above.

In the second scheme, for the Poisson equation, superconvergent points are all symmetry points. However, for the Laplace equation, some non-symmetry points are also superconvergent points. For instance, when $n = 2$, the function value superconvergent points for the Poisson equation are the vertices and the midpoints of diagonal edges; for the Laplace equation, the midpoints of horizontal and vertical edges are superconvergent points as well. Note that the midpoints of edges parallel to the axes are not symmetry points. Similarly, for derivatives, superconvergence only happens at the second order Gaussian points of the diagonal edges for the Poisson equations, while it occurs also at the Gaussian points of the edges parallel to the axes for the Laplace equation.

4. Lagrange Pentahedral element

4.1. Determining $\Phi_{n+1}(K)\backslash V_n(K)$ and $\Phi^{\mathcal{H}}_{n+1}(K)\backslash V_n(K)$. Consider Lagrangian pentahedral elements. We shall note that, in this case, the nth order FE space $V_n(K)$ consists of basis functions of degree greater than n. In particular,

$$\Pi_{n+1}(K) \setminus V_n(K) = \text{Span}\left\{\zeta^{n+1},\ \xi^{n+1-i}\eta^i,\ i = 0, \ldots, n+1\right\},$$

whose dimension is $(n + 3)$. Then $\Phi_{n+1}(K)\backslash V_n(K)$ can be determined by the way described in Section 3.

We take the case of $n = 3$ as an example. To simplify notations, we write piecewisely defined function

$$f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{in} \quad e_1 \\ f_2(\mathbf{x}) & \text{in} \quad e_2 \end{cases} \tag{4.14}$$

as

$$f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{cases}. \tag{4.15}$$

For monomial ζ^4, we have $I_n(\zeta^4) = \frac{6}{5}\zeta^2 - \frac{1}{5}$, where I_n is similar as the interpolation operator described in §3.1. Solving the FE problem (3.9), we get the correction term $z^\pi|_{\zeta^4} = 0$. Thus, the basis function in $\Phi_4(K)$ associated to ζ^4 is

$$\zeta^4 - I_n(\zeta^4) - z^\pi|_{\zeta^4} = \zeta^4 - \frac{6}{5}\zeta^2 + \frac{1}{5} = \frac{8}{5}\,\phi_4(\zeta). \tag{4.16}$$

As another example, consider monomial $\xi^3\eta$. The interpolation is obtained as

$$I_n(\xi^3\eta) = \begin{cases} -\xi^3 + \xi^2\eta + \xi^2 + \frac{1}{5}\xi\eta + \frac{1}{5}\xi - \frac{1}{5}\eta - \frac{1}{5} \\ \xi^3 - \xi^2\eta + \xi^2 + \frac{1}{5}\xi\eta - \frac{1}{5}\xi + \frac{1}{5}\eta - \frac{1}{5} \end{cases}. \tag{4.17}$$

From (3.9), we get the correction term

$$(4.18)\qquad z^\pi|_{\xi^3\eta} = \begin{cases} \frac{5}{14}\xi^2\eta - \frac{5}{14}\xi\eta^2 + \frac{3}{14}\xi^2 - \frac{4}{7}\xi\eta + \frac{3}{14}\eta^2 - \frac{3}{14}\xi + \frac{3}{14}\eta + \frac{1}{21} \\ -\frac{5}{14}\xi^2\eta + \frac{5}{14}\xi\eta^2 + \frac{3}{14}\xi^2 - \frac{4}{7}\xi\eta + \frac{3}{14}\eta^2 + \frac{3}{14}\xi - \frac{3}{14}\eta + \frac{1}{21} \end{cases}.$$

Thus, the basis function in $\Phi_4(K)$ associated to $\xi^3\eta$ is $\xi^3\eta - I_n(\xi^3\eta) - z^\pi|_{\xi^3\eta}$, which is independent to ζ. Moreover, we shall note that it is a basis function for the triangular elements in regular pattern studied in [14].

Apply the process to all monomials in $\Pi_{n+1}(K)\setminus V_n(K)$, we get a basis of $\Phi_4(K)\backslash V_3(K)$.

$\psi_4^1 = \phi_4(\zeta)$,

$$\psi_4^2 = \begin{cases} \xi^4 - \frac{3}{7}\xi^2\eta + \frac{3}{7}\xi\eta^2 - \frac{51}{35}\xi^2 + \frac{24}{35}\xi\eta - \frac{9}{35}\eta^2 + \frac{9}{35}\xi - \frac{9}{35}\eta + \frac{1}{7} \\ \xi^4 + \frac{3}{7}\xi^2\eta - \frac{3}{7}\xi\eta^2 - \frac{51}{35}\xi^2 + \frac{24}{35}\xi\eta - \frac{9}{35}\eta^2 - \frac{9}{35}\xi + \frac{9}{35}\eta + \frac{1}{7} \end{cases},$$

$$\psi_4^3 = \begin{cases} \xi^3\eta + \xi^3 - \frac{19}{14}\xi^2\eta + \frac{5}{14}\xi\eta^2 - \frac{17}{14}\xi^2 + \frac{13}{35}\xi\eta - \frac{3}{14}\eta^2 + \frac{1}{70}\xi - \frac{1}{70}\eta + \frac{16}{105} \\ \xi^3\eta - \xi^3 + \frac{19}{14}\xi^2\eta - \frac{5}{14}\xi\eta^2 - \frac{17}{14}\xi^2 + \frac{13}{35}\xi\eta - \frac{3}{14}\eta^2 - \frac{1}{70}\xi + \frac{1}{70}\eta + \frac{16}{105} \end{cases},$$

$$\psi_4^4 = \begin{cases} \xi^2\eta^2 + \frac{6}{7}\xi^2\eta - \frac{6}{7}\xi\eta^2 + \frac{1}{21}\xi^2 - \frac{116}{105}\xi\eta + \frac{1}{21}\eta^2 - \frac{26}{105}\xi + \frac{26}{105}\eta + \frac{43}{315} \\ \xi^2\eta^2 - \frac{6}{7}\xi^2\eta + \frac{6}{7}\xi\eta^2 + \frac{1}{21}\xi^2 - \frac{116}{105}\xi\eta + \frac{1}{21}\eta^2 + \frac{26}{105}\xi - \frac{26}{105}\eta + \frac{43}{315} \end{cases},$$

$$\psi_4^5 = \begin{cases} \xi\eta^3 - \frac{5}{14}\xi^2\eta + \frac{19}{14}\xi\eta^2 - \eta^3 - \frac{3}{14}\xi^2 + \frac{13}{35}\xi\eta - \frac{17}{14}\eta^2 + \frac{1}{70}\xi - \frac{1}{70}\eta + \frac{16}{105} \\ \xi\eta^3 + \frac{5}{14}\xi^2\eta - \frac{19}{14}\xi\eta^2 + \eta^3 - \frac{3}{14}\xi^2 + \frac{13}{35}\xi\eta - \frac{17}{14}\eta^2 - \frac{1}{70}\xi + \frac{1}{70}\eta + \frac{16}{105} \end{cases},$$

$$\psi_4^6 = \begin{cases} \eta^4 - \frac{3}{7}\xi^2\eta + \frac{3}{7}\xi\eta^2 - \frac{9}{35}\xi^2 + \frac{24}{35}\xi\eta - \frac{51}{35}\eta^2 + \frac{9}{35}\xi - \frac{9}{35}\eta + \frac{1}{7} \\ \eta^4 + \frac{3}{7}\xi^2\eta - \frac{3}{7}\xi\eta^2 - \frac{9}{35}\xi^2 + \frac{24}{35}\xi\eta - \frac{51}{35}\eta^2 - \frac{9}{35}\xi + \frac{9}{35}\eta + \frac{1}{7} \end{cases}.$$

We note that, any polynomial in $\mathcal{H}_4(K)\setminus V_3(K)$ is a combination of ζ^4, $\xi^4 - 6\xi^2\eta^2 + \eta^4$, and $\xi^3\eta - \xi\eta^3$. Accordingly, $\Phi_4^{\mathcal{H}}(K)\backslash V_3(K)$ has a basis of ψ_4^1, $\psi_4^2 - 6\psi_4^4 + \psi_4^6$, and $\psi_4^3 - \psi_4^5$.

4.2. Superconvergence Results. As mentioned in Section 3, we can study superconvergence of the Poisson and the Laplace equations.

It is an important fact that ψ_4^2, ψ_4^3, ψ_4^4, ψ_4^5, and ψ_4^6 form exactly a basis of the polynomial space $\Phi_4(K)$ for the regular triangular element specified in [14], which is independent of ζ. Consequently, the superconvergent points for Lagrangian pentahedral elements are obtained from the tensor-product of the one-dimensional superconvergent points in ζ-direction and the superconvergent points for regular triangular elements in $\xi\eta$-plane. See Section 6. We shall also declare that this is a general fact for the Lagrangian pentahedral elements.

5. Serendipity pentahedral element

5.1. Superconvergence for the Poisson Equation. For serendipity pentahedral elements, we have

$$\Pi_{n+1}(K)\setminus V_n(K) = \mathrm{Span}\left\{\xi^{n+1-i-j}\eta^i\zeta^j \mid i,j,i+j = 0,\ldots,n+1,\ j \neq 1,n\right\},$$

which has dimension 4 for $n = 1$ and dimension $n(n+3)/2$ for $n > 1$. Using the process introduced in Section 3 to all monomials in $\Pi_{n+1}(K)\setminus V_n(K)$, we can determine $\Phi_{n+1}(K)\backslash V_n(K)$ accordingly. We shall note that, when $n = 1$ and 2, the spaces for serendipity elements are the same as those for Lagrangian elements.

When $n \geq 3$, the spaces are different. For instance, $\Phi_4(K)\backslash V_3(K)$ has 9 basis functions, where the first 6 are the same as those for Lagrangian elements specified in §4.1. The others are

$$\psi_4^7 = \tilde{\phi}_2(\xi)\tilde{\phi}_2(\zeta),$$
$$\psi_4^8 = \tilde{\phi}_2(\eta)\tilde{\phi}_2(\zeta),$$
$$\psi_4^9 = \begin{cases} \xi\eta\zeta^2 + \xi\zeta^2 - \eta\zeta^2 - \frac{1}{3}\xi\eta - \frac{2}{3}\zeta^2 - \frac{1}{3}\xi + \frac{1}{3}\eta + \frac{2}{9} \\ \xi\eta\zeta^2 - \xi\zeta^2 + \eta\zeta^2 - \frac{1}{3}\xi\eta - \frac{2}{3}\zeta^2 + \frac{1}{3}\xi - \frac{1}{3}\eta + \frac{2}{9} \end{cases}.$$

where $\tilde{\phi}_2 = \phi_2 + \frac{1}{3}$.

Hence, for 3rd order serendipity pentahedral elements, the function value superconvergent points of the Poisson equation are common zeros of ψ_4^i, $i = 1, \ldots, 9$. It is straightforward to check that there is not any common zero. Therefore there is no function value superconvergent point in this case.

On the other hand, the derivative superconvergent points are common zeros of the corresponding derivative of the basis functions. When $n = 3$, the ξ- and η-derivative superconvergence results for the serendipity element are the same as those for the Lagrangian element. But the ζ-derivative superconvergent points are only on the plane $\{\zeta = 0\}$, rather than on all the Gaussian planes $\{P_3(\zeta) = 0\}$ for the Lagrangian element.

5.2. Superconvergence for the Laplace Equation. We next consider superconvergence of the Laplace equation. Note that $\xi\zeta^3$, $\eta\zeta^3$, and $\xi^i\eta^{3-i}\zeta$, $i = 0, \ldots, 3$, are all in the local FE space $V_3(K)$. Thus, a harmonic polynomial in $\mathcal{H}_4(K)\backslash V_3(K)$ is a combination of the following 5 functions, instead of the 9 basis functions in $\mathcal{H}_4(K)$.

$$\begin{gathered} \xi^4 - 6\xi^2\eta^2 + \eta^4, \quad \eta^4 - 6\eta^2\zeta^2 + \zeta^4, \quad \zeta^4 - 6\zeta^2\xi^2 + \xi^4, \\ \xi^3\eta - \xi\eta^3, \qquad \xi^3\eta - 3\xi\eta\zeta^2. \end{gathered} \tag{5.19}$$

Therefore, $\Phi_4^{\mathcal{H}}(K)\backslash V_3(K)$ has a basis of

$$\begin{gathered} \psi_4^1 - 6\psi_4^7 + \psi_4^2, \quad \psi_4^1 - 6\psi_4^8 + \psi_4^6, \quad \psi_4^2 - 6\psi_4^4 + \psi_4^6, \\ \psi_4^3 - \psi_4^5, \qquad \psi_4^3 - 3\psi_4^9. \end{gathered} \tag{5.20}$$

Thus, the 3rd order function value superconvergent points are common zeros of the 5 functions in (5.20). Similar as for the Poisson equation, there is no superconvergent point for the Laplace equation in this case.

The derivative superconvergent points are common zeros of the corresponding derivative of the basis functions. For cubic element, the ξ- and η-derivative superconvergence results for the Laplace equation are the same as those for the Poisson equation. However, it is straightforward to verify that, the ζ-derivative superconvergent points for the Laplace equation are on the plane $\{\zeta = 0\}$ and the points $\pm(1 - \sqrt{3}/3, -1 + \sqrt{3}/3, \pm\sqrt{90 - 50\sqrt{3}}/5)$, which are different from the results for the Poisson equation.

6. Summary

In this section, we gather the superconvergence results for the tetrahedral, Lagrangian pentahedral, and serendipity pentahedral elements of both the Poisson and the Laplace equations. Function value and derivative superconvergence for elements of order 2 and 3 are involved in the following tables.

TABLE 1. FUNCTION VALUE SUPERCONVERGENCE RESULTS

	Tetra.	**Lagrangian Penta.**		**Serendipity Penta.**	
n	**PE, LE**	**PE**	**LE**	**PE**	**LE**
2	SPK	SPK	SPK $\cup$ Set1	SPK	SPK $\cup$ Set1
3	None	None	Set2	None	None

Here, PE and LE stand for the "Poisson equation" and the "Laplace equation", respectively. SPK is the set of the 27 "symmetry points of K"

$$\{(\xi,\eta,\zeta) \mid \xi,\eta,\zeta=-1,0, \text{ or } 1\}.$$

Set1 is the set of 12 points in the tensor product

$$\left\{\pm\left(\tfrac{1}{4},-\tfrac{1}{4}\right)\pm\tfrac{\sqrt{7}}{4}(1,1)\right\} \otimes \{\pm 1, 0\},$$

and Set2 is the set of 48 points in

$$\left\{\pm\left(\pm\tfrac{\sqrt{225-30\sqrt{30}}}{15},1\right),\ \pm\left(1,\pm\tfrac{\sqrt{225-30\sqrt{30}}}{15}\right),\ \pm(1,1)\cdot\tfrac{\sqrt{450\pm30\sqrt{105}}}{30}\right\}\otimes\left\{\pm1,\pm\tfrac{1}{\sqrt{5}}\right\}.$$

Recall that the ξ- and η-derivative superconvergent points for the pentahedral elements are symmetric. Therefore, we collect only the ξ- and ζ-derivative superconvergence results for the pentahedral elements in Table 2.

TABLE 2. DERIVATIVE SUPERCONVERGENCE RESULTS

	Tetra.	**Lagrangian Penta.**		**Serendipity Penta.**	
n	**PE,LE**	**PE**	**LE**	**PE**	**LE**
2	GPt	ξ- : Set3 ζ- : PL$_2$	ξ- : Set3 $\cup$ Set4 ζ- : PL$_2$	ξ- : Set3 ζ- : PL$_2$	ξ- : Set3 $\cup$ Set4 ζ- : PL$_2$
3	MPt	ξ- : Set5 ζ- : PL$_3$	ξ- : Set6 ζ- : PL$_3$	ξ- : Set5 ζ- : $\{\zeta=0\}$	ξ- : Set6 ζ- : Set7

Here, GPt (resp. MPt) stands for the set of "second order Gaussian points (resp. midpoints) of the edges parallel to the corresponding derivative direction"; PL$_n$ is the set of Gaussian planes $\{P_n(\zeta)=0\}$. Moreover, we have

Set3: lines $\left\{\xi=\pm\frac{1}{\sqrt{3}},\eta=\pm1\right\}$;

Set4: lines $\left\{\xi=\frac{1}{2},\eta=-\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right\}$ and $\left\{\xi=-\frac{1}{2},\eta=\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right\}$;

Set5: lines $\{\xi=0,\eta=\pm1\}$;

Set7: plane $\{\zeta=0\}$ and points $\left\{\pm\left(1-\frac{\sqrt{3}}{3},\ -1+\frac{\sqrt{3}}{3},\ \pm\frac{\sqrt{90-50\sqrt{3}}}{5}\right)\right\}$,

and Set6 is the union of the lines

$$\begin{aligned}&\{\xi=0,\ \eta=\pm1\}\ \cup\ \{\xi=\pm1,\ \eta=\pm1\}\\ \cup&\left\{256\xi^5-416\xi^4+173\xi^3+21\xi^2-17\xi-1=0,\ \eta=\pm\sqrt{3\xi^2-2\xi},\ \xi\ge\eta\right\}\\ \cup&\left\{256\xi^5+416\xi^4+173\xi^3-21\xi^2-17\xi+1=0,\ \eta=\pm\sqrt{3\xi^2+2\xi},\ \xi\le\eta\right\}.\end{aligned}$$

The reader is referred to [20] for the superconvergence results of tetrahedral and pentahedral elements of order up to 4 and 5, respectively. We stop pursuing superconvergent points of higher order elements after the first internal mode is involved. However, one may apply this method to obtain superconvergence results for a desired element without essential difficulty.

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College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081 (China), and Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

E-mail: zzhang@math.wayne.edu

URL: http://www.math.wayne.edu/∼zzhang/

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

E-mail: linr@math.wayne.edu

URL: http://www.math.wayne.edu/∼linr/